Newton's Method: A Mathematical Model of Giant Swings

An Honors Paper for the Department of Mathematics

Introduction

What is a Giant Swing?

In gymnastics, giant swings are one of the fundamental tricks on both the woman's uneven parallel bars and the men's high bar. They serve as a skill in themselves but are more commonly used as a transitional element that generates power for more difficult tricks. A gymnast begins her swing in a handstand on top of the bar, then falling stomach first, completes a full circle around the bar in a stretched out position until she returns to handstand. In fact, there are two giant swing techniques currently taught, dictating the gymnast's body position between these key points. The traditional technique is considered more conventional and still is typical in women's gymnastics. A gymnast using this technique keeps her shoulders open throughout the entire swing and pumps her legs at the bottom of the swing. The skill develops as follows,

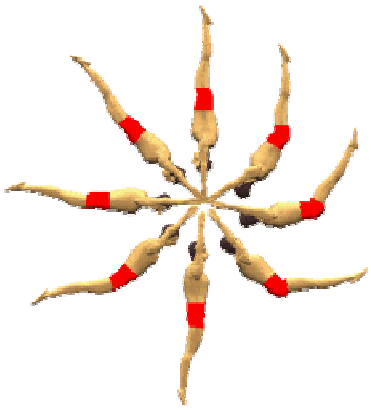


Figure (i)
(David Kerwin, England)

In contrast, the power technique has recently become rather common in men's gymnastics, under the justification that it creates a more substantial increase in power thus allowing for the execution of more difficult skills. This method involves using both the shoulder and hip angles throughout the swing. A progression appears as follows,

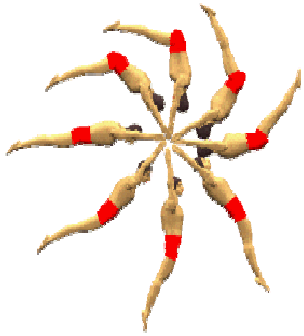


Figure (ii)
(David Kerwin, England)

Note that these techniques are quite distinct. Yet the literature is still indecisive on the relative merits of each. Though studies have addressed the issue of maximum overall power from a variety of angles, it has not been conclusively shown that the power technique is any more effective than the traditional method.

In fact, previous modes of investigation ought to be noted here. Giant swings have been studied though a number of biomechanical approaches. From a physical standpoint, force on the bar and muscle strain of the gymnast have been considered. Video analysis has offered insights on angular velocity and the particularity of giant swings followed by other skills. Models have been developed largely from the standpoint of either moments of inertia or Lagrangian mechanics. These models incorporate a wide range of variables, including moments of inertia, hinge points, friction, bar elasticity, and constraints presented by a low bar. While this information is certainly instructive, here we will attempt a generally more abstract discussion, that perhaps is more closely related to the pendulum models as opposed to physical data. We seek a physically reasonable model that incorporates parameters that we can work with in order to satisfy particular initial and endpoint conditions. That is, we would like to develop a reasonable model that starts our gymnast in a handstand on the bar and returns her to the same position at a particular endpoint time.

The Approach

Before embarking on our analysis we ought to map out our general path. As the pendulum is perhaps the fundamental example from which a model for giant swings can be derived, it will be our point of reference along the way. To begin our investigation,

we will consider a method of creating models using physics, specifically Newtonian and Lagrangian Dynamics. With this procedure established, we develop a few basic pendulum models and consider the modes in which these models might lead to a model for our swinging gymnast. The issue of energy conservation will be addressed, prompting the inclusion of forcing in the model. Furthermore, the threat of chaos will become a underlying concern. This presents the issue of choice of forcing function; what function, and parameters within the function, are desirable? We approach this question through Newton's method. But before applying Newton's method, in Section II we recall the theory of differential equations necessary for its employment. Section II also brings with it a final physical model which is developed further in Section III. Once we have clearly presented our question, in Section III we also amend the procedure of Newton's method so as to make it appropriate to our particular question. In Section IV, Newton's method is used to approach the simple forced pendulum with friction model. Multiple forcing functions are compared and the tenuous nature of Newton's Method is noted. In fact, we are forced to further refine our procedure and still fall short of a final solution.

A Note on Chaos

Throughout our discussion we will frequently encounter chaos as a lurking presence in our dynamical system. While a full development of chaos theory is not included, here we will briefly present most basic concepts of chaos so as to ground our later references. Further discussion of chaos can be found in Baker and Gollub, Chaotic Dynamics.

Very generally, a chaotic system is one in which two solutions beginning incredibly near each other end up quickly taking very different paths. That is, the solution is said to be sensitive to small perturbations. Therefore, chaotic systems are particularly concerning in the physical sciences. Given the instrumental error associated with any measurement, accurate long term predications are unattainable if the system model is chaotic.

It is important to note that chaos is possible only in systems of dimension three or greater. This condition is perhaps best demonstrated with reference to the phase plane. Briefly, the phase plane is a parametric plot of solution curves to a system of differential equations. Axes are given by each of the dependant variables, where any explicit time

dependence is also considered to be a dimension itself. Arranged as such, solution curves ought not cross in the phase plane. Yet is sensitivity to initial conditions possible under this restriction; can two solutions that begin very close to each other diverge quickly (exponentially)? In fact, this can occur only through solutions folding back on themselves and creating a three dimensional chaotic attractor with an infinite number of layers (see Baker and Gollub). Hence the necessity of at least three dimensions for a system displaying chaotic properties.

There are additional characteristics that many chaotic systems hold. Often chaotic systems are self similar. That is, magnification of a plot of a chaotic system leads to a plot that is qualitatively alike to the original. Chaotic systems also tend to have very irregular behavior. A stroboscopic observation of a chaotic system (as is the procedure in Poincaré sections) shows no consistency of position over time intervals. Behavior of chaotic systems can also be well represented through a variety of expressions. The logistic map displays many of the most basic properties of a chaotic pendulum, the circle map demonstrates the phase locking phenomena, and the difference equation provides a simple yet physically instructive model.

Furthermore, a few tools have been developed in attempt to "measure" chaos; that is, to help determine the relative sensitivity of various systems or a particular system under various parameter values. A variety of dimensions have been defined, including the correlation dimension, the information dimension, and the metric dimension, each of which provide a particular measure of the chaos of a system. Lyapunov exponents and Kolmogorov entropy are two other quantities defined to aid in the analysis of chaotic systems. Finally, bifurcation diagrams are of service in the study of chaos relative to parameter values. While we do not employ any of these modes of analysis, further studies could take this route and would most likely shed considerable light on our project.

Section I

In order to establish the physics necessary to construct a mathematical model of the swinging gymnast, we begin in the context of a more basic system, namely the simple pendulum. Newtonian mechanics will serve as our initial means to a system of equations. The principle of least action will then be developed, leading us to the derivation of Lagrange's equation. This alternative method to procuring equations of motion will then be applied to the same simple pendulum system and contrasted with Newtonian mechanics.

Newtonian Mechanics

Newtonian Mechanics is the typical approach to basic physical systems. Here we will attempt to develop equations of motion for the simple pendulum system using Newton's second law, namely:

$$F = ma \tag{1.1}$$

First, let's consider the pertinent parameters and variables in the simple pendulum model. θ, the variable function of t, is the angle measured counterclockwise positive where vertically down is zero. The pendulum arm is mass less of length l and the bob is a point mass with mass m, as in Figure 1.1 below.

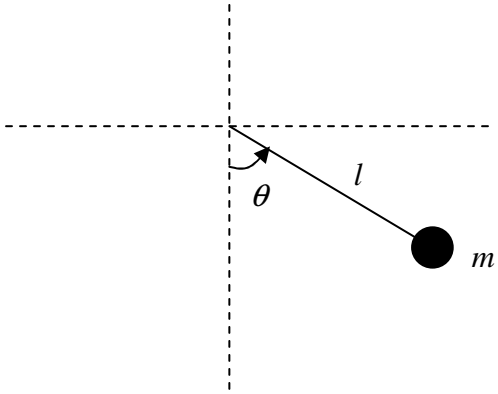


Figure 1.1

From here, the Cartesian position of the bob can be expressed as

$$(l\sin\theta(t), l\cos\theta(t)) \tag{1.2}$$

From this representation it is clear that the angular velocity of the bob is $l\frac{d\theta}{dt}$ and the angular acceleration is $l\frac{d^2\theta}{dt^2}$. In order to use Newton's second law, we must also establish the forces acting on the bob. Gravity and friction are our two primary concerns. Gravity acts as

$$-mg\sin\theta \tag{1.3}$$

And friction acts as

$$-bl\frac{d\theta}{dt} \tag{1.4}$$

where *b* is the coefficient of friction, a constant greater than zero. It is interesting to note that this is a reasonable, but only one of many possible choices of friction functions. Here, friction acts against the angular velocity in proportion to it. While this is certainly a feasible representation, one could argue that in the swinging gymnast situation, friction has a larger effect at the bottom of the swing, when the gymnast must tighten her grip on the bar in order not to fall off, and less of an effect at the top of the swing. Thus eventually we might consider a more sophisticated frictional term of the sort $b\cos\frac{d\theta}{dt}$.

But, returning to our original representation of friction and using (1.1) we get

$$\frac{d^2\theta}{dt^2}+\frac{b}{m}\frac{d\theta}{dt}+\frac{g}{l}\sin\theta=0 \tag{1.5}$$

which, taking $\frac{d\theta}{dt}=\dot{\theta}$ can also be represented as a system of equations:

$$\begin{aligned}\frac{d\theta}{dt}&=\dot{\theta}\\ \frac{d\dot{\theta}}{dt}&=-\frac{b}{m}\dot{\theta}-\frac{g}{l}\sin\theta\end{aligned} \tag{1.6}$$

Thus if we can establish the forces acting on our system it is possible to use Newton's second law to procure equations of motion. Yet for more complicated systems, precisely establishing the forces is exceedingly difficult. Hence we ought to consider an another approach, namely Lagrangian Dynamics.

Lagrangian Dynamics

Lagrangian Dynamics serve as an alternative means to procuring equations of motion for physical system. While Newtonian Mechanics focuses on establishing forces acting within a system, Lagrangian Dynamics requires the determination of energies of a system. Using the calculus of variation and the principle of least action we can establish Lagrange's equation which can be applied to the system at hand to obtain the equations of motion.

Lagrange's equation is best understood in the context of the principle of least action, or Hamilton's Principle. Hamilton's Principle states:

> Of all the possible paths along which a dynamical system may move from one point to another in configuration space within a specified time interval, the actual path followed is that which minimizes the time integral of the Lagrangian function for the system. (Thornton, p. 237)

Clearly, this statement warrants some unraveling. Broadly speaking, given a starting and ending arrangement of a system, Hamilton's Principle specifies the way the system will get from the first to second. The Lagrangian function of a system is the difference between the kinetic and the potential energies, given by

$$L(q_i, \dot{q}_i, t) = T(q_i, \dot{q}_i, t) - U(q_i, \dot{q}_i, t) \tag{2.1}$$

where T is kinetic energy, U is potential energy, q_i is a set of generalize coordinates and $\dot{q}_i$ are their time derivatives. Hence Hamilton's principle puts forth the basic question of the calculus of variation. We would like to determine the function such that the expression,

$$J = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)dt \, . \tag{2.2}$$

is at an extreme value. In words, expression (2.2) states that our solution function q_i is such that taking any other neighboring function $\overline{q}_i$ will cause *J* to increase. Let's define these neighboring functions more precisely. We do this by parameterization with α. Let

$$q_i = q_i(\alpha, t) \quad \text{such that} \quad q_i = q_i(0, t) = q_i(t) \tag{2.3}$$

then clearly $q_i(t)$ is our original function that minimizes J. Then we can say,

$$q_i(\alpha, t) = q_i(0, t) + \alpha\eta(t) \tag{2.4}$$

where our function $\eta(t)$ has a continuous first derivative and $\eta(t_1) = \eta(t_2) = 0$. The second condition is necessary because the endpoints of our family of functions are fixed—we are searching for the one function, starting at one particular point and going to another fixed point, that minimizes the integral over the path. Our parameterized integral is thus,

$$J = \int_{t_1}^{t_2} L(q_i(\alpha,t), \dot{q}_i(\alpha,t), t)dt. \tag{2.5}$$

As we would like an extreme value, in other words, a stationary value on our parameterized curve, we must have,

$$\frac{\partial J}{\partial \alpha}|_{\alpha=0} = 0 \tag{2.6}$$

for all functions $\eta(t)$. Taking this partial derivative, we have

$$\frac{\partial J}{\partial \alpha} = \frac{\partial}{\partial \alpha}\int_{t_1}^{t_2} L(q_i(\alpha,t), \dot{q}_i(\alpha,t), t)dt = \int_{t_1}^{t_2} \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \alpha}dt\,. \tag{2.7}$$

Recalling equation (2.4), we also have

$$\begin{aligned} \frac{\partial q_i}{\partial \alpha} &= \eta(t) \\ \frac{\partial \dot{q}_i}{\partial \alpha} &= \frac{d\eta}{dt} \end{aligned} \tag{2.8}$$

which when used in (2.7) results in the expression,

$$\frac{\partial J}{\partial \alpha} = \int_{t_1}^{t_2} \frac{\partial L}{\partial q_i}\eta(t) + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \eta}{\partial t}dt\,. \tag{2.9}$$

If we use integration by parts on the second term on the right hand side we get,

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \eta}{\partial t}dt = \frac{\partial L}{\partial \dot{q}_i}\eta(t)|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\eta(t)dt\,. \tag{2.10}$$

Since $\eta(t_1) = \eta(t_2) = 0$, we have,

$$\frac{\partial J}{\partial \alpha} = \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\eta(t)dt\,. \tag{2.11}$$

Recall by (2.6) that this expression is equal to zero. Note that the integral seems to be independent of α, though really q_i and $\dot{q}_i$ are parameterized to have dependence on α. Yet for the particular α value $\alpha = 0$ we know this expression must equal zero. Also note

that $\eta(t)$ is an arbitrary function, so we cannot use it to make the expression equal to zero. Therefore, the expression in parentheses must itself equal zero. Thus we have Euler's equation, or since we have applied this analysis to our particular problem and Hamilton's principle, we get Lagrange's equation, the result

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \tag{2.12}$$

Hence each generalized coordinate has a resulting equation. These equations and a set of initial conditions fully describe the manner in which the system moves. Let's consider their application to our basic example of the simple pendulum.

Lagrangian Dynamics and the Simple Pendulum

First, we chose generalized coordinates to represent our system. In fact, in this instance, it is wise to set up the system in the same manner as was used when applying Newton's second law. Hence our sole coordinate is θ. Starting from the Cartesian representation of the coordinates of the bob, using the well established expressions for kinetic and potential energy, and putting them in terms of θ, we get for the energies and Lagrangian function

$$T = \frac{1}{2} ml^2 \dot{\theta}^2 \tag{3.1}$$

$$U = -mgl\cos(\theta) \tag{3.2}$$

$$L(\theta, \dot{\theta}, t) = T - U = \frac{1}{2} ml^2 \dot{\theta}^2 + mgl\cos(\theta) \tag{3.3}$$

Doing out the partial and time derivates and using (2.12) we get

$$ml^2 \ddot{\theta} + mgl\sin\theta = 0 \tag{3.4}$$

Which can be written as the system:

$$\frac{d\theta}{dt} = \dot{\theta} \tag{3.5}$$

$$\frac{d\dot{\theta}}{dt} = -\frac{g}{l}\sin\theta \tag{3.6}$$

Comparing this system to the system derived Newtonian mechanics (see (1.6)) we see that they are very similar. In fact, we might add on a dissipative frictional term to the

system at this point, recalling the form it ought to take from our knowledge of Newtonian mechanics. The system then becomes:

$$\frac{d\theta}{dt} = \dot{\theta} \tag{3.7}$$

$$\frac{d\dot{\theta}}{dt} = -\frac{b}{m}\dot{\theta} - \frac{g}{l}\sin\theta \tag{3.8}$$

This is identical to the system we derived using Newtonian mechanics (see (1.6)) and in fact, it can be shown that the conclusions drawn from the Newtonian and Lagrangian approaches are inherently equivalent (see Marion and Thornton).

Hamiltonians

Now that we have established the physics necessary to discuss our swinging gymnast, it is important to note a few mathematical concepts seen in the simple pendulum system that will be of use later in our final model of the swinging gymnast. Through the discussion of Hamiltonian functions, we are led to conversation about the phase plane and energy conservation.

Let's start by defining what it means to be Hamiltonian.

Definition: *If we have a system of equations*

$$\frac{dx}{dt} = F(x, y) \tag{4.1}$$

$$\frac{dy}{dt} = G(x, y) \tag{4.2}$$

that satisfies the condition

$$\frac{\partial F}{\partial x} = -\frac{\partial G}{\partial y} \tag{4.3}$$

then the system is Hamiltonian. Furthermore, there exists a function H such that

$$F(x, y) = \frac{\partial H}{\partial y} \tag{4.4}$$

$$G(x, y) = -\frac{\partial H}{\partial x}. \tag{4.5}$$

Let's consider this definition in the context of our system of equations derived for the simple pendulum with friction ((3.7) and (3.8)). We'll idealize this pendulum by taking the coefficient of friction to be zero. Our system then becomes:

$$\frac{d\theta}{dt} = \dot{\theta} \tag{4.6}$$

$$\frac{d\dot{\theta}}{dt} = -\frac{g}{l}\sin\theta \tag{4.7}$$

Proposition: *This system has a Hamiltonian function.*

First, let's see if the system satisfies (4.3). We have,

$$\frac{\partial}{\partial\theta}\dot{\theta} = 0 = -\frac{\partial}{\partial\dot{\theta}}\left(-\frac{g}{l}\sin\theta\right). \tag{4.8}$$

Thus the Hamiltonian condition has been fulfilled. Our next logical move is to attempt to figure out the Hamiltonian function. As a first guess, let's take the total energy of the system. Let

$$H(\theta,\dot{\theta}) = T(\theta,\dot{\theta}) + U(\theta,\dot{\theta}) \tag{4.9}$$

be the total energy function. Then in the instance of the simple pendulum without friction, we have

$$H(\theta,\dot{\theta}) = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos(\theta) \tag{4.10}$$

Consider the partial derivates of $H(\theta,\dot{\theta})$. We have

$$\frac{\partial H}{\partial\theta} = mgl\sin(\theta) \tag{4.11}$$

$$\frac{\partial H}{\partial\dot{\theta}} = ml^2\dot{\theta}\,. \tag{4.12}$$

If the total energy was in fact our Hamiltonian function, then we would expect $-\frac{\partial H}{\partial\theta}$ to be identical to our second equation of motion, namely (4.7), and $\frac{\partial H}{\partial\dot{\theta}}$ to be identical to our first equation of motion, namely (4.6). Clearly this is not the case, but the similarity is striking. In fact, our function seems to be off by a constant factor of ml^2. Hence, our Hamiltonian function must be $\frac{H(\theta,\dot{\theta})}{ml^2}$.

As we have shown, our simple pendulum system is Hamiltonian, with Hamiltonian function closely related to the total energy (i.e. the sum of the kinetic and potential energies of the system). But there are further properties of Hamiltonian functions that we ought to note. First of all, it can be shown that in two dimensional Hamiltonian systems, equilibrium points will be centers or saddle points or there will be

only one $\lambda = 0$ eigenvalue for the system. Additionally, as we would like to learn about the change in energy of the system throughout the swing (namely, our goal is to have an increase in energy) we might consider the time derivate of our total energy function. As the total energy function is a constant multiple of the Hamiltonian function, their behaviors in this context are equivalent. Consider

$$\frac{dH}{dt} = \frac{\partial H}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial H}{\partial \dot{\theta}}\frac{d\dot{\theta}}{dt} \tag{4.13}$$

which using our expressions for $\frac{d\theta}{dt}$ and $\frac{d\dot{\theta}}{dt}$ as given by out system (4.6) and (4.7) becomes

$$\frac{dH}{dt} = 0\,. \tag{4.14}$$

Thus the time derivative of the Hamiltonian function is zero. Or—perhaps a more interesting way to put it for our simple pendulum system—the total energy of the system does not change as time goes on. Given an initial energy for the system, the motion of the system will be such that the energy does not change with time.

Proposition: *This in fact is true of all Hamiltonian system.*

Using Lagrangian mechanics the way we did to derive these equations, we will always end up with a system in which energy is conserved as will be later shown. Unfortunately, this is not ideal in the context of our swinging gymnast model. But first, more practically, we ought to consider the phase plane along with the implications of the time derivative of the Hamiltonian equaling zero.

The Phase Plane/Space

Phase space provides us a way of visualizing solutions to our system of equations. An n-dimensional system will have n-dimensional phase space with each dimension corresponding to one of the variables. For our simple pendulum, we discuss the phase plane with θ and $\dot{\theta}$ as our axes. Each point in phase space corresponds to a possible initial condition for our system. Through each point passes a curve parameterized in time, that traces out the trajectory of θ and $\dot{\theta}$ as time passes. Due to uniqueness of solutions, we know each initial condition (i.e. each point in phase space) can have only one solution, thus curves in phase space cannot cross (as long as any time dependence in

the original system is taken as an additional dimension). Let's consider phase space for our simple pendulum.

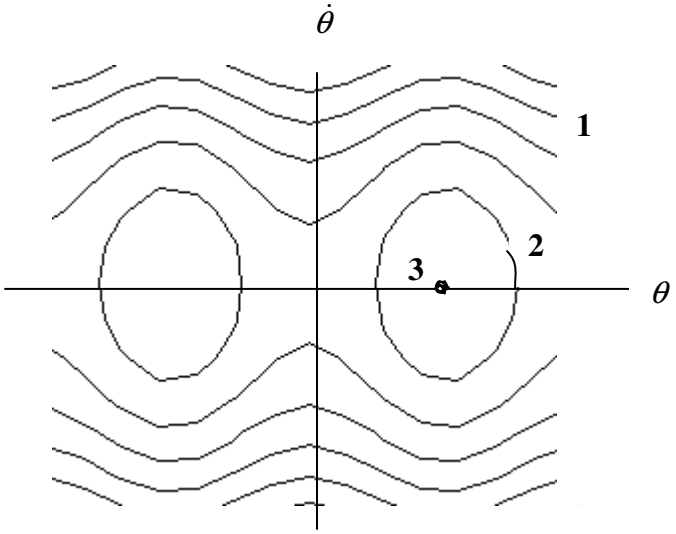


Figure 6.1

Note that the equilibrium points are centers and saddles, as we knew to expect from the observation that the system was Hamiltonian. What other observations can be made here? Recall that each curve traces out a solution for the system given a particular initial condition. Also, we noted that since the system is Hamiltonian, given any initial condition, the total energy does not change. Hence, we can conclude that each solution curve in the phase plane corresponds to a system with a given energy. In fact, three types of behavior can be noted from the phase plane. The curve labeled three in the phase plane above, corresponds to a pendulum that is never displaced from its equilibrium point. Hence there is no change from the initial position as time passes and this corresponds to the lowest possible energy. The curve labeled two above corresponds to what we consider to be typical pendulum back and forth motion. Clearly as we travel along the curve, the magnitude of θ will increase as the magnitude of $\dot{\theta}$ decreases or vice versa. This curve has higher energy than curve one. The highest energy of the labeled curves is curve one. This corresponds most directly to our desired swinging gymnast motion. As time passes, the angle of the pendulum continues to increase, as would be the case for a gymnast successfully swinging around the bar. In sum, phase space can help us determine a variety of solution behaviors. Additionally, given a Hamiltonian system, the solution curves in phase space and the level sets of the

Hamiltonian function will be identical. Having oriented ourselves in phase space, we ought to return to the issue of conservation of energy and attempt to gain some sort of insight into predicting the occurrence of Hamiltonian functions.

Lagrangian Dynamics Implies Conservation of Energy

Conservation of energy is clearly an important question for our model. Furthermore, we've see that Lagrangian mechanics builds equations by starting with expressions for the energies. In fact, we find that by analyzing the theory behind Lagrangian mechanics we reach an important result about the total energy.

We begin with a few observations.

Observation One:

$$L - \sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j = -H. \tag{6.1}$$

This statement warrants justification. As our system is closed, the Lagrangian cannot depend explicitly on time. That is,

$$\frac{\partial L}{\partial t} = 0. \tag{6.2}$$

Take the total derivative of the Lagrangian. We have,

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j} \dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j \tag{6.3}$$

which, using Lagrange's equations, namely $\frac{\partial L}{\partial q_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}$, becomes

$$\frac{dL}{dt} = \sum_j \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} \dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j. \tag{6.4}$$

Rearranging these terms we have,

$$\frac{d}{dt}\left(L - \sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j \right) = 0. \tag{6.5}$$

So the total time derivative of the expression is zero, or in other words, this expression is constant in time. Naming this constant $-H$, we have,

$$L - \sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j = -H. \tag{6.6}$$

Observation Two: *The potential energy does not depend explicitly on* $\dot{q}_j$ *or time and our generalized coordinates are "nicely" related to our rectangular coordinates.*

While these are fairly self explanatory statements, their consequences are significant. The partial derivative of the potential energy with respect to $\dot{q}_j$ will be zero. Therefore, we have,

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial (T-U)}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}. \tag{6.7}$$

Using this with equation (6.6) results in the expression,

$$(T-U) - \sum_j \frac{\partial T}{\partial \dot{q}_j}\dot{q}_j = -H. \tag{6.8}$$

We would like to be able to simplify the summation expression.

Observation Three:

$$\sum_l \dot{q}_l \frac{dT}{d\dot{q}_l} = 2\sum_{k,l} a_{lk}\dot{q}_k\dot{q}_l = 2T \tag{6.9}$$

This statement ought to be justified. Note that in the instances we will be considering, the kinetic energy is a function of the square of the velocities and does not explicitly depend on time. Hence, (see Thornton and Marion)

$$T = \sum_{j,k} a_{jk}\dot{q}_j\dot{q}_k \tag{6.10}$$

Differentiating with respect to $\dot{q}_l$ gives,

$$\frac{dT}{d\dot{q}_l} = \sum_k a_{lk}\dot{q}_k + \sum_j a_{jl}\dot{q}_j \tag{6.11}$$

Let's now multiply by $\dot{q}_l$ and sum over *l*. We have,

$$\sum_l \dot{q}_l \frac{dT}{d\dot{q}_l} = \sum_{k,l} a_{lk}\dot{q}_k\dot{q}_l + \sum_{j,l} a_{jl}\dot{q}_j\dot{q}_l \,. \tag{6.12}$$

Yet note that the similarity each term on the right hand side bears to (6.10). In fact, they are conceptually identical and thus we have,

$$\sum_l \dot{q}_l \frac{dT}{d\dot{q}_l} = 2\sum_{k,l} a_{lk}\dot{q}_k\dot{q}_l = 2T\,. \tag{6.13}$$

Therefore, returning to the expression (6.8), we can now state,

$$\begin{aligned}(T-U)-2T &= -H \\ T+U &= H\end{aligned} \tag{6.14}$$

Recall that *H* is a constant. So in fact we have shown that total energy of the system is constant. In other words, when we derive equations of motion using Lagrangian dynamics as we have, energy is always going to be conserved. We will later find that forcing and dissipative functions present a manner of eliminating energy conservation.

What Happened to Friction?

Note that in the previous discussion we have elected to set the coefficient of friction *b* equal to zero, in effect, ignoring the influence of friction on the system. Yet friction is clearly applicable to our swinging gymnast, hence we ought to consider the ways in which including it alters models in general. We have established that by using Lagrangian Dynamics conservation of energy is guaranteed. What happens if we first derive equations of motion using Lagrangian Dynamics, and then affix to those equations frictional term? Is energy still conserved in our system? How are the solution curves in the phase plane affected? What sort of long term behaviors can we expect?

First, let's consider the issue of energy conservation. Common sense physics would indicate that as friction generates heat and heat is a way in which energy can be "lost" we might expect that by including a frictional term in our system of equations the change in energy over time would be negative.

Proposition: *Affixing a frictional term to a system generated with Lagrangian mechanics causes* $\frac{dH}{dt} < 0$.

Let's see if this hypothesis holds up under example. Consider once again our simple pendulum system, but now we will take the coefficient of friction, *b*, to be nonzero. Thus we have

$$\frac{d\theta}{dt} = \dot{\theta} \tag{7.1}$$

$$\frac{d\dot{\theta}}{dt} = -\frac{b}{m}\dot{\theta} - \frac{g}{l}\sin\theta \tag{7.2}$$

Using the chain rule again,

$$\frac{dH}{dt} = \frac{\partial H}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial H}{\partial \dot{\theta}}\frac{d\dot{\theta}}{dt} = -bml^2\dot{\theta}^2 \tag{7.3}$$

Note that as the constants *b* and *m* are defined to be nonnegative and the other two components are squared, the negative sign here guarantees that $\frac{dH}{dt} < 0$ always. Thus the simple pendulum with friction model demonstrates to us that the introduction of a frictional term to a system that was previously conservative results in a system that loses energy over time.

Definition: *A function L(θ, $\dot{\theta}$) is a Lyapunov function if for every solution* $(\theta(t), \dot{\theta}(t))$ *that is not an equilibrium solution,* $\frac{d}{dt}L(\theta(t), \dot{\theta}(t)) \le 0$ *for all t with strict inequality at all but a discrete set of t values.*

With the previous example so close at hand, we can't help but note that *H* serves as a Lyapunov function for the single pendulum with friction system, as for any solution $\frac{dH}{dt} < 0$ for all *t* values. This observation provides a significant insight to the behavior of solutions in the phase plane. *H* is the same energy function that was found in the simple pendulum without friction model, but now the level sets of *H* play a different role. We know that the solution curves of our system will trace out a path in the phase plane from higher to lower energy, since we have shown the system must be losing energy over time. The level sets of *H* each correspond to a particular energy, so while they are not the solution curves to our system, they point us in the generally correct direction. We know our solution curves must travel in a roughly similar manner as the level sets of *H* while crossing them from higher to lower energies. Hence we might draw in some possible solution curves below, superimposed on the level sets of the Hamiltonian.

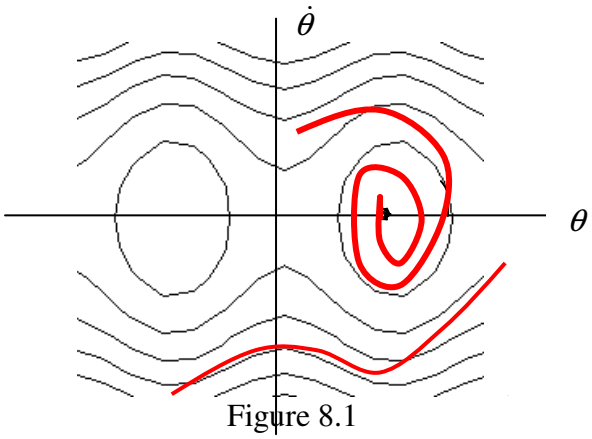


Figure 8.1

It is important to note the long term behavior of such dissipative systems. As all our solution curves are perpetually crossing from higher to lower energies, eventually all our curves will get sucked into a low energy equilibrium point. As we noted earlier, the low energy equilibrium points are those on the θ axis. Hence on the long term, no matter what our initial condition is, our simple pendulum with friction system will end up with zero angular velocity, sitting at some angle $2\pi n$ ($n \in \square$). In other words, the pendulum will come to a resting vertically downward position, having possibly completed some number of full revolutions. Thus in regards to our swinging gymnast, friction seems to be an unfortunate development. When we begin from any system derived using Lagrangian Dynamics, combining in a frictional term will cause the system to lose energy over time, resulting in all solutions tending to the "rest position." In observing these behaviors in the simple pendulum model we know we must develop a way to preemptively adjust our more sophisticated final model so that it becomes possible to accurately represent the gain in energy of the swinging gymnast.

Simple Mass Distribution Pendulum

In this vein, we might also consider the affects of switching our point mass pendulum to a pendulum with mass distribution along the arm. Consider our new physical set up:

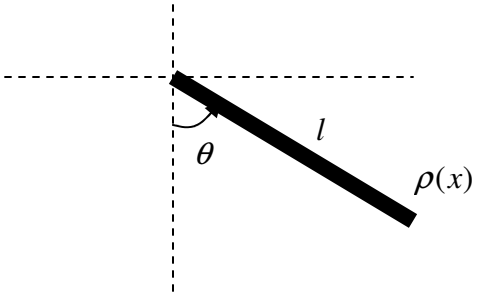


Figure 9.1

Here instead of a massless arm with a point mass bob of mass *m*, the arm of the pendulum has mass distribution $\rho(x)$ where x is taken as the position along the arm starting from the origin and going to *l*, the length of the arm. What are the energies of this system? Consider the potential and kinetic energy of any given point mass along the pendulum arm. We have

$$U = -\rho(x) g x \cos\theta \tag{8.1}$$

$$T = \frac{1}{2}\rho(x) x^2 \dot{\theta}\,. \tag{8.2}$$

Integrating this over the length of the pendulum arm to find the total energy, we get

$$U = -g\cos\theta \int_0^l x\rho(x)dx \tag{8.3}$$

$$T = \frac{1}{2}\dot{\theta}\int_0^l x^2\rho(x)dx\,. \tag{8.4}$$

Note that the constant in the potential energy expression is the first moment of the pendulum arm, and the constant in the kinetic energy is the second moment. Renaming these M_1 and M_2 we have:

$$L(\theta,\dot{\theta}) = \frac{1}{2}M_2\dot{\theta}^2 + gM_1\cos\theta\,. \tag{8.5}$$

Recall the Lagrangian function for the simple pendulum:

$$L(\theta,\dot{\theta}) = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos(\theta). \tag{8.6}$$

These two are notably similar. In fact, if we can identify M_2 with ml^2 and M_1 with ml then the two would be identical and as our equations of motion are built from the Lagrangian, they too would be equivalent. Is this identification possible? We would have

$$M_1 = \int_0^l x\rho(x)dx = ml \tag{8.7}$$

$$M_2 = \int_0^l x^2 \rho(x) dx = ml^2 \tag{8.8}$$

Indeed, there is nothing inconsistent that would prohibit both of these equations from holding true. Our new point mass pendulum ought to have mass $m = \frac{M_1^2}{M_2}$ and length $l = \frac{M_2}{M_1}$. Hence making wise choices of parameters values for our simple point mass pendulum we can match up the energies so that they correspond to that of a mass distribution pendulum. Thus the equations of motions of the two are equivalent under a renaming of constants. Therefore working with a mass distribution simple pendulum instead of a point mass simple pendulum is an unnecessary complication. Unfortunately, we will not be quite as lucky in the instance of later models. Once a hinge point is added to the pendulum, there is a qualitative difference in behavior between the point mass pendulum and the mass distribution pendulum.

Section II

Later analysis of our swinging gymnast model will use Newton's method in attempt to find parameter values that produce a certain solution behavior for our system. Yet this procedure requires justification; in order for us to use Newton's method we need a system with solutions that are continuous with respect to the parameters. Also the solutions to the system must have continuous second partial derivatives with respect to the parameters. Therefore, we ought to establish the theory behind these two properties before proceeding further with our model. Under what conditions will Newton's method be a functional tool? We first establish the basic definitions and theorems through proofs of existence and uniqueness of solutions. From there we consider the theorems necessary for Newton's method.

Theory: Existence of Solutions

The first step to solving our system of differential equations is confirming that there is in fact a solution. First we precisely define our system of equations, with prime (') denoting a total time derivative. We have,

$$(t, \varphi(t)) \qquad (t \in I) \tag{1.1}$$

$$\varphi'(t) = f(t, \varphi(t)) \qquad (t \in I). \tag{1.2}$$

In finding a solution of this system, we seek a differential function $\varphi(t)$ that satisfies the above. More plainly, $\varphi(t)$ is a solution if there exists a t interval I on which $\varphi(t)$ satisfies our differential equation

$$x' = f(t, x). \tag{1.3}$$

Yet, phrased as such our question is incomplete. What additional information is required to fully establish this system? In fact, by prescribing an initial value for the system of equations, we can hope to eventually establish a unique solution to our problem. Our question is now appended by an initial condition. We seek a solution $\varphi(t)$ such that

$$x(\tau) = \xi \tag{1.4}$$

$$x' = f(t, x). \tag{1.5}$$

With that established, we can consider the existence of a solution. The Cauchy-Peano Existence Theorem states:

Theorem A1 (Cauchy - Peano) : *If* $f \in C$ *on the rectangle R then* $\exists$ *a solution* $\varphi \in C^1$ *of* $x' = f(t, x)$ *on* $|t-\tau| \le \alpha$ *for which* $\varphi(\tau)=\xi$

Let's unpack this some. Given an initial condition, we can draw a rectangle, call it R, bounded on the *t* axis by $\tau - \alpha$ and $\tau + \alpha$. We have the figure,

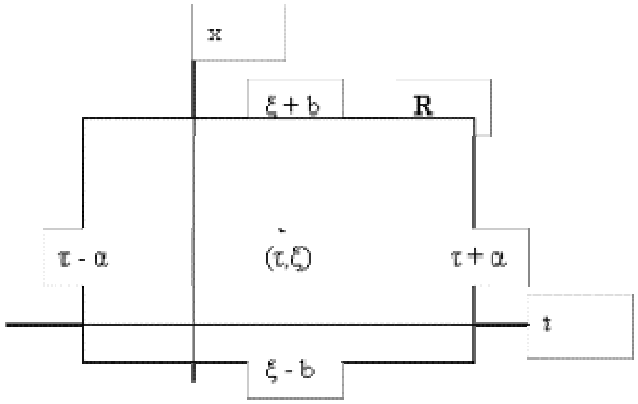


Figure 1.1

If the right hand side of our system of differential equations is continuous, then we know that within R the system has a solution with a continuous first derivative. In order to prove this theorem, we rely on the idea of ε-approximate solutions and the Ascoli Lemma.

Definition: *An ε-approximate solution satisfies the conditions,*

$$\begin{aligned} &(i)\ (t, \varphi(t)) \in D \quad (t \in I) \\ &(ii)\ \varphi \in C^1 \text{ on I, except for a finite set of points S on I} \\ &\qquad \text{where } \varphi' \text{ can have simple discontinuities} \\ &(iii)\ |\varphi'(t) - f(t, \varphi(t))| \le \varepsilon \quad (t \in I - S) \end{aligned} \tag{1.6}$$

In other words, an ε-approximate solution is a continuous function defined on a *t* interval *I* such that the solution is part of the domain in the (t, x) plane, the solution is first derivative continuous except at a finite set of points where there may be simple discontinuities (call this set *S*), and finally the absolute value of the difference between the first derivative of the solution and the right hand side of the system of differential equations is less than or equal to ε on *I* - *S*.

By another theorem, we know that as long as the right hand side of the system is continuous, there exists an ε-approximate solution on our rectangle R that satisfies our initial condition. These solutions are constructed in a piecewise manner as follows:

$$\begin{aligned} &\varphi(\tau) = \xi \\ &\varphi(\tau) = \varphi(t_{k-1}) + f(t_{k-1}, \varphi(t_{k-1}))(t - t_{k-1}) \\ &t_{k-1} < t \le t_k \qquad k = 1, 2...., n \end{aligned} \tag{1.7}$$

Here we begin by satisfying the initial condition and from there build a linear solution on the next subinterval by using the system of equations to approximate the appropriate slope. From there, the end point of the previous interval is used to dictate the starting position and slope for the new interval. By continuing to construct line segments in this manner, we end up with an ε-approximate solution on the entire interval.

The Ascoli Lemma is another important tool necessary for this proof. It says

Lemma A2 (Ascoli) : *On a bounded interval* I, *let* $F = \{f\}$ *be an infinite, uniformly bounded, equicontinuous* set *of functions. Then F contains a squence* $\{f_n\}$, n=1, 2, ... *which is uniformly convergent on* I.

Thus under certain conditions we can say that a infinite subset of a set of functions is uniformly convergent on a bounded interval. This lemma in fact allows us to apply the idea of ε-approximate solutions towards creating an actual solution of our system of equations. Before doing this though we ought to fully understand the necessary conditions by recalling the definition of equicontinuity:

Definition: *A set of function is equicontinuous on a t interval if given any* $\varepsilon > 0$ *there exists a* $\delta > 0$ *independent of f, t, and* $\tilde{t}$ *such that* $|f(t) - f(\tilde{t})| < \varepsilon$ *whenever* $|t - \tilde{t}| < \delta$.

In other words, equicontinuity extends our definition of uniform continuity to a whole set of functions. For a given ε the set of functions are all uniformly continuous under the same δ value. With this condition defined, let's consider the use of the Ascoli Lemma in proving the existence of solutions.

Consider $\{\varepsilon_n\}$ (n = 1, 2…) a monotonically decreasing set of positive real number tending to zero. Then for each of these ε_n we can find a corresponding ε_n-approximate solution to our system of equations. Create a set of these solutions, $\{\varphi_n\}$. It can be shown the this set is uniformly bounded and equicontinuous. Therefore, we can use the Ascoli Lemma and say that the sequence of ε-approximate solutions has a uniformly convergent subsequence. Call this subsequence $\{\varphi_{nk}\}$ and the limit function it converges to φ. As all the φ_{nk} are continuous they must be converging to a continuous

limit function, that is φ must be continuous. Furthermore, by considering the way in which we go about building the ε-approximate solutions, and noting that $f(t,\varphi_{nk}(t)) \to f(t,\varphi(t))$ uniformly, it can be shown that the limit function φ is in fact the desired solution to the system and satisfies our initial condition.

Therefore we have established the conditions for existence of solutions to our system of differentiable equations. However, we should note that we have only established solution existence on a bounded interval.

Theory: Uniqueness of Solutions

Having verified the existence of solutions, we would like to know if such solutions are unique.

Theorem B1*: Let $f \in (C, Lip)$ in D and $(\tau,\xi) \in D$. If φ_1 and φ_2 are any two solutions of x'=f(t, x) on (a, b), $a < \tau < b$ such that $\varphi_1(\tau) = \varphi_2(\tau) = \xi$ then $\varphi_1 = \varphi_2$.*

This theorem follows from a particular case of an established inequality. Before outlining this inequality and its use, we ought to understand the conditions of this theorem. D is the domain in the *(t, x)* plane on which *f*, the right hand side of our system of equations, is defined. Our initial condition is inside this domain and we are considering solutions defined on a particular bounded interval including our initial condition. *f* is also continuous and Lipschitz. Recall the definition of Lipschitz:

Definition: *A function f is Lipschitz if there exists a constant k > 0 such that for every (t,x_1) and (t,x_2) in the domain, $|f(t,x_1) - f(t,x_2)| \le k|x_1 - x_2|$.*

Given two x values a certain distance apart, the corresponding distance between the function values will be less than a constant multiple of the x distance, where the proportionality constant applies for all distances and all x values.

Lemma B2: *If f is differentiable with respect to its components then f is Lipschitz.*

The higher dimensional case will be taken here as it is ultimately the one of concern. As *f* is differentiable and taken on a particular compact interval, there exists a maximum value of *f'*, call this value *P*. Assume there exists an x_1 and x_2 such that $\frac{\|f(t,x_1)-f(t,x_2)\|}{\|x_1-x_2\|} = P_1 > P$. Then by the mean value theorem, as *f'* is continuous, there must

exist an x_3 between x_1 and x_2 such that $f'(t, x_3) = P_1$. Yet, as P is the maximum value of f' on this interval, this cannot be the case. So we can take P to be our Lipschitz constant, as we know $\frac{\|f(t,x_1)-f(t,x_2)\|}{\|x_1-x_2\|} < P$ for all x_1 and x_2 in our interval, and thus $\|f(t,x_1)-f(t,x_2)\| < P\|x_1-x_2\|$. So we have shown that differentiability on an interval is sufficient to guarantee that a function is Lipschitz on that interval. We use this fact later in confirming that our particular systems of equations are Lipschitz. With this understood, we can now take on the inequality necessary for this proof.

Perhaps it is best to start with the inequality itself and then work to understand its parts. The inequality is as follows (see Coddington and Levinson for proof):

$$|\varphi_1(t)-\varphi_2(t)| \le \delta e^{k|t-\tau|} + \frac{\varepsilon}{k}(e^{k|t-\tau|}-1) \qquad (2.1)$$

Here, $\varphi_1(t)$ and $\varphi_2(t)$ are ε_1 and ε_2-approximate solutions respectively. ε is the sum of ε_1 and ε_2. k is the Lipschitz constant. τ is such that $|\varphi_1(\tau)-\varphi_2(\tau)| \le \delta$ where τ is also within the interval that the solutions are defined on. Now let's apply this inequality towards proving our theorem. Here φ_1 and φ_2 are in fact actual solutions to the system. Hence ε_1 and ε_2 both equal zero. So (2.1) becomes

$$|\varphi_1(t)-\varphi_2(t)| \le \delta e^{k|t-\tau|} \qquad (2.2)$$

Furthermore, since φ_1 and φ_2 both satisfy the same initial condition, $|\varphi_1(\tau)-\varphi_2(\tau)| = 0$, δ can be taken to be zero. So we have, $|\varphi_1(t)-\varphi_2(t)| \le 0$ which implies that $\varphi_1(t) = \varphi_2(t)$, that is the solutions are unique. Therefore, by considering a particular application of our inequality (2.1), we have established a theorem for the uniqueness of solutions on a bounded interval.

Theory: Continuation of Solutions

In the past few sections, we have established the existence and uniqueness of solutions on a bounded interval. We would now like to know more about the nature of the solutions in general. If we know a solution exists on a particular interval, when will it exist on a larger interval? We consider this question using continuation of solutions.

First, let's cite the pertinent theorem.

Theorem C1: *Let $f \in C$ in a domain D of the (t, x) plane, and suppose f is bounded on D. If φ is a solution of x'=f(x, t) on an interval (a, b) then the limits φ(a+0) and φ(b-0) exist. If (a, φ(a + 0)) [or (b, φ(b - 0))] is in D, then the solution of φ may be continued to the left of a (or to the right of b).*

To restate this, if the limits of the solution at the endpoints are in the domain, then our solution can be continued through an interval passed the endpoint, or in other words, we can find a solution to the system of equations on a larger interval.

Now let's consider the general idea of the proof. The existence of these limits is established through using the Cauchy criterion for convergence while recalling that $\varphi(t) = \zeta + \int_{\tau}^{t} f(s, \varphi(s))ds \quad (t \in (a,b))$. With these limits established, we can consider the continuation of the solution through using the endpoint limit. What we are doing here is defining a new initial value problem using the same system to attempt to continue the solution over a larger interval. If the endpoint limit is in the domain, then we can take that value to be the new initial condition. By our existence theorem, we know a solution must exist that satisfies this initial condition. Furthermore if *f* is Lipschitz on the domain, then the solution for the new initial condition is unique and there is only one possible continuation of our original solution. Therefore, given a solution on a particular interval, whenever the limit of the solution values at the endpoints exist as part of the domain, the solution can be continued over a larger interval.

In the course of establishing the existence and uniqueness of solutions, we have constructed the basic framework necessary for any theoretical study of systems of differentiable equations. Furthermore, we have developed some of the key tool, such as the Lipschitz condition, which are necessary for understanding the proofs particular to Newton's method. The particular theorems we require for Newton's method are continuity of solutions relative to parameters and the continuity of the partial derivatives of the solution. We now discuss these theorems and their proofs.

Continuity of Solutions with Respect to Parameters

Theorem D1: *If $x' = f(t, x, \mu)$, where f is continuous on the Cartesian product of a region R in (t, y) space where $a < t < b$ and a region $I_{\mu} = \{\mu : \|\mu - a\| < c, c \in \mathbb{R}_{>0}, a \in k \text{ vector}\}$*

in parameter space, and f is also Lipschitz with Lipschitz constant k, then a solution of x' = f(t, x, μ) depends continuously on the parameter.

If we are to hope to find zeros of a function, as in Newton's method, we need to establish that between a "positive" and "negative" solution, exists a zero solution. Continuity serves as a sufficient condition here; hence the necessity of this theorem. We consider continuity with respect to parameters as that is the fashion in which we will employ Newton's method.

Now consider the proof. We would like to show that for all $\varepsilon > 0$ there exists a δ such that if two parameter vectors are within δ of each other then the solutions will be within ε of each other. That is, $\|\mu_1 - \mu_0\| < \delta$ implies that $\psi_1 \in K(\psi_0, \varepsilon)$.

We define our δ as follows,

$$\delta < \varepsilon e^{-k(b-a)}. \tag{4.1}$$

and take ψ_0 to be the solutions of $x' = f(t, x, \mu_0)$ and ψ_1 the solution of $x' = f(t, x, \mu_1)$ both under the initial condition (τ, b). Therefore, we can write,

$$\psi_0(t) = b + \int_\tau^t f(s, \psi_0(s), \mu_0) ds \tag{4.2}$$

$$\psi_1(t) = b + \int_\tau^t f(s, \psi_1(s), \mu_1) ds. \tag{4.3}$$

Also note that by the continuity of *f* with respect to the parameters, we can pick a δ such that

$$\|f(t, \psi_0(t), \mu_1) - f(t, \psi_0(t), \mu_0)\| < \frac{\varepsilon}{b-a} e^{-k(b-a)} \tag{4.4}$$

for all *t* in [a, b] where $\|\mu_1 - \mu_0\| < \delta$. Subtracting the two representations of our solutions, we have,

$$\|\psi_0(t) - \psi_1(t)\| \le \int_\tau^t \|f(s, \psi_0(s), \mu_0) - f(s, \psi_1(s), \mu_1)\| ds. \tag{4.5}$$

Upon employing the triangle inequality this gives,

$$\|\psi_0(t) - \psi_1(t)\| \le \int_\tau^t \|f(s, \psi_0(s), \mu_0) - f(s, \psi_0(s), \mu_1)\| ds + \int_\tau^t \|f(s, \psi_0(s), \mu_1) - f(s, \psi_1(s), \mu_1)\| ds. \tag{4.6}$$

Note that each term on the right hand side can be simplified. Using (4.4) to simplify the first term (note that $t - \tau < b - a$) and the fact that *f* is Lipschitz to simplify the second term we have,

$$\|\psi_0(t)-\psi_1(t)\| \le \varepsilon e^{-K(b-a)} + \int_\tau^t K\|\psi_0(s)-\psi_1(s)\|ds. \tag{4.7}$$

In fact, this expression can be rewritten in an illuminating manner using the replacement,

$$R(t) = \int_\tau^t K\|\psi_0(s)-\psi_1(s)\|ds \tag{4.8}$$

and noting our original representation of δ. Thus we can represent (4.7) as,

$$\frac{1}{K}R'(t) \le \delta + R(t). \tag{4.9}$$

Manipulating this expression by multiplying by Ke^{-Kt} and rearranging terms, we have

$$R'(t)e^{-Kt} - R(t)Ke^{-Kt} \le \delta e^{-Kt}. \tag{4.10}$$

Considering the left hand side of this equation and recalling that our formation of *R(t)* implies that $R(\tau) = 0$, we notice that this expression is the derivative of $R(t)e^{-Kt}$. So integrating both sides of this equation from τ to t gives us,

$$R(t)e^{-Kt} \le -\delta e^{-Kt} + \delta e^{-K\tau} \tag{4.11}$$

$$\delta + R(t) \le \delta e^{K(t-\tau)}. \tag{4.12}$$

Using our relationship in (4.9), we can state,

$$\frac{1}{K}R'(t) \le \delta e^{K(t-\tau)} \tag{4.13}$$

which becomes, when returning to our definition of *R(t)*, and recalling our relationship between δ and ε,

$$\|\psi_0(t)-\psi_1(t)\| \le \delta e^{K(t-\tau)} \le \delta e^{K(b-a)} < \varepsilon. \tag{4.14}$$

Hence we have shown,

$$\|\psi_0(t)-\psi_1(t)\| < \varepsilon \tag{4.15}$$

which in turn implies, since $\|\psi_0 - \psi_1\| = \sup_{t\in[a,b]} \|\psi_0(t)-\psi_1(t)\|$, that $\psi_1 \in K(\psi_0, \varepsilon)$ as long as $\|\mu_1 - \mu_0\| < \delta$. Thus we have shown that our solution depends continuously on the parameters. The corollary follows,

Corollary D2: *Consider x' = f(t, x, μ) as above. Then there exists a unique solution ψ of x' = f(t, x, μ) on $a \le t \le b$ satisfying our initial condition and moreover, ψ is continuous on the (n+k+2)-dimensional domain.*

This statement is stronger in that it gives continuity of solutions in additional directions. That is, the solutions are continuous not only in the parameter direction or the initial condition direction, but along a "diagonal" of these directions. For example, in terms of Newton's method this implies that we could still expect to find zeros even if we chose to alter parameter values and initial condition values simultaneously. It is helpful to have this possibility at hand.

A Proof About Determinants

An alternative method of the continuity of solutions with respect to parameters proof relies on a property of determinants. While we stick with the proof above, we ought to consider this fairly basic and rather useful property of determinants. See Coddington and Levinson for its particular use in the continuity proof.

Theorem E1: *if φ_{ij} is a component of the matrix φ with variable t, with prime (') indicating the derivative with respect to t, then we have,*

$$(\det\varphi)' = \begin{vmatrix} \varphi_{11}' & \varphi_{11}' & \cdots & \varphi_{1n}' \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_{n1} & \varphi_{n2} & \cdots & \varphi_{nn} \end{vmatrix} + \begin{vmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1n} \\ \varphi'_{21} & \varphi'_{22} & \cdots & \varphi'_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_{n1} & \varphi_{n2} & \cdots & \varphi_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1n} \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \varphi'_{n1} & \varphi'_{n2} & \cdots & \varphi'_{nn} \end{vmatrix}$$

Proof through induction.

Base case: consider $\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

The left hand side of the theorem becomes: $(\det\varphi)' = (ad - bc)' = ad' + a'd - bc' - b'c$.

Considering the right hand side of the theorem we have:

$$\begin{vmatrix} a' & b' \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ c' & d' \end{vmatrix} = a'd - b'c + ad' - bc' = ad' + a'd - bc' - b'c = (\det\varphi)'.$$

So the theorem holds for the n = 2 case.

Assume the theorem holds for n = k case φ_k.

Prove it holds for the n = k +1 case φ_{k+1}.

From the basic properties of determinants, we know

$$\det\varphi_{k+1}=\varphi_{11}\begin{vmatrix}\varphi_{22}&\cdots&\varphi_{2(k+1)}\\ \vdots&\ddots&\vdots\\ \varphi_{(k+1)2}&\cdots&\varphi_{(k+1)(k+1)}\end{vmatrix}-\varphi_{12}\begin{vmatrix}\varphi_{21}&\varphi_{23}&\cdots&\varphi_{2(k+1)}\\ \vdots&\vdots&\ddots&\vdots\\ \varphi_{(k+1)2}&\varphi_{(k+1)3}&\cdots&\varphi_{(k+1)(k+1)}\end{vmatrix}+\cdots\pm\varphi_{1(k+1)}\begin{vmatrix}\varphi_{21}&\cdots&\varphi_{2k}\\ \vdots&\ddots&\vdots\\ \varphi_{(k+1)2}&\cdots&\varphi_{(k+1)k}\end{vmatrix}$$

Where the last sign is positive if *k+1* is even and negative if *k+1* is odd. Using the chain rule to differentiate, we have,

$$(\det\varphi_{k+1})'=\varphi_{11}'\begin{vmatrix}\varphi_{22}&\cdots&\varphi_{2(k+1)}\\ \vdots&\ddots&\vdots\\ \varphi_{(k+1)2}&\cdots&\varphi_{(k+1)(k+1)}\end{vmatrix}+\varphi_{11}\begin{vmatrix}\varphi_{22}&\cdots&\varphi_{2(k+1)}\\ \vdots&\ddots&\vdots\\ \varphi_{(k+1)2}&\cdots&\varphi_{(k+1)(k+1)}\end{vmatrix}'-\varphi_{12}'\begin{vmatrix}\varphi_{21}&\varphi_{23}&\cdots&\varphi_{2(k+1)}\\ \vdots&\vdots&\ddots&\vdots\\ \varphi_{(k+1)2}&\varphi_{(k+1)3}&\cdots&\varphi_{(k+1)(k+1)}\end{vmatrix}-$$
$$\varphi_{12}\begin{vmatrix}\varphi_{21}&\varphi_{23}&\cdots&\varphi_{2(k+1)}\\ \vdots&\vdots&\ddots&\vdots\\ \varphi_{(k+1)2}&\varphi_{(k+1)3}&\cdots&\varphi_{(k+1)(k+1)}\end{vmatrix}'+\cdots\pm\varphi_{1(k+1)}'\begin{vmatrix}\varphi_{21}&\cdots&\varphi_{2k}\\ \vdots&\ddots&\vdots\\ \varphi_{(k+1)2}&\cdots&\varphi_{(k+1)k}\end{vmatrix}\pm\varphi_{1(k+1)}\begin{vmatrix}\varphi_{21}&\cdots&\varphi_{2k}\\ \vdots&\ddots&\vdots\\ \varphi_{(k+1)2}&\cdots&\varphi_{(k+1)k}\end{vmatrix}' \qquad (5.1)$$

Note that our matrixes here are now *k* by *k* dimensional. Therefore, when taking the determinant then the derivative of them, we can apply our theorem as we have assumed it to be true for the $n = k$ case. Also note the first, third, and fifth terms in the expression. When taken with the remaining terms existing within the dots, they create the determinant of a *(k+1)* by *(k+1)* matrix with first row derivative. Using this fact and applying our theorem to the *k* by *k* matrixes we have,

$$(\det\varphi_{k+1})'=\begin{vmatrix}\varphi_{11}'&\varphi_{12}'&\cdots&\varphi_{1(k+1)}'\\ \varphi_{21}&\varphi_{22}&\cdots&\varphi_{2(k+1)}\\ \vdots&\vdots&\ddots&\vdots\\ \varphi_{(k+1)1}&\varphi_{(k+1)2}&\cdots&\varphi_{(k+1)(k+1)}\end{vmatrix}+\varphi_{11}\begin{vmatrix}\varphi_{22}'&\cdots&\varphi_{2(k+1)}'\\ \vdots&\ddots&\vdots\\ \varphi_{(k+1)2}&\cdots&\varphi_{(k+1)(k+1)}\end{vmatrix}+\ldots+\varphi_{11}\begin{vmatrix}\varphi_{22}&\cdots&\varphi_{2(k+1)}\\ \vdots&\ddots&\vdots\\ \varphi_{(k+1)2}'&\cdots&\varphi_{(k+1)(k+1)}'\end{vmatrix}$$
$$-\varphi_{12}\begin{vmatrix}\varphi_{21}'&\varphi_{23}'&\cdots&\varphi_{2(k+1)}'\\ \vdots&\vdots&\ddots&\vdots\\ \varphi_{(k+1)2}&\varphi_{(k+1)3}&\cdots&\varphi_{(k+1)(k+1)}\end{vmatrix}-\ldots-\varphi_{12}\begin{vmatrix}\varphi_{21}&\varphi_{23}&\cdots&\varphi_{2(k+1)}\\ \vdots&\vdots&\ddots&\vdots\\ \varphi_{(k+1)2}'&\varphi_{(k+1)3}'&\cdots&\varphi_{(k+1)(k+1)}'\end{vmatrix}+\cdots$$
$$\pm\varphi_{1(k+1)}\begin{vmatrix}\varphi_{21}'&\cdots&\varphi_{2k}'\\ \vdots&\ddots&\vdots\\ \varphi_{(k+1)2}&\cdots&\varphi_{(k+1)k}\end{vmatrix}\pm\ldots\pm\varphi_{1(k+1)}\begin{vmatrix}\varphi_{21}&\cdots&\varphi_{2k}\\ \vdots&\ddots&\vdots\\ \varphi_{(k+1)2}'&\cdots&\varphi_{(k+1)k}'\end{vmatrix}$$

(5.2)

Again, we can condense many of these terms. Take the second, fourth, and sixth terms. When placed with the proper terms within the dots, they make up the determinant of a *(k+1)* by *(k+1)* matrix with derivative in the second row. Similarly, the third, fifth, and seventh terms lead to the determinant of the *(k+1)* by *(k+1)* matrix with derivative in the

final row. The determinants of matrixes with derivatives in the rows between are contained within the dots. We have,

$$(\det \varphi_{K+1})' = \begin{vmatrix} \varphi_{11}' & \varphi_{12}' & \cdots & \varphi_{1(k+1)}' \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2(k+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{(k+1)1} & \varphi_{(k+1)2} & \cdots & \varphi_{(k+1)(k+1)} \end{vmatrix} + \begin{vmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1(k+1)} \\ \varphi_{21}' & \varphi_{22}' & \cdots & \varphi_{2(k+1)}' \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{(k+1)1} & \varphi_{(k+1)2} & \cdots & \varphi_{(k+1)(k+1)} \end{vmatrix} + \ldots + \begin{vmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1(k+1)} \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2(k+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{(k+1)1}' & \varphi_{(k+1)2}' & \cdots & \varphi_{(k+1)(k+1)}' \end{vmatrix}$$

(5.3)

Thus we have shown the property to be true for the $n = k+1$ case, and by induction it holds true in general. With this theorem proven, we conclude our interesting side note, and return to discuss the final theorem necessary for Newton's method: continuity of the partial derivatives of the solution.

Continuity of Partial Derivatives

For our particular use of Newton's method we will eventually require that the partial derivatives of the solution with respect to the parameters be second derivative continuous in parameter and initial condition space. We will see that this is necessary to guarantee the convergence of Newton's method provided we start within some δ of our solution parameter vector. What conditions are necessary here?

Theorem F1: *Consider $x' = f(t, x, \mu)$ and its solution $\varphi(t, x, \mu)$, as above with the existence of continuous f_x and f_μ. Then $\frac{\partial \varphi}{\partial \mu}$ is a continuous solution of $y' = f_x(t, \varphi(t, \tau, \xi, \mu), \mu)\, y + f_\mu(t, \varphi(t, \tau, \xi, \mu), \mu)$.*

Proof of this statement can be found in Coddington and Levinson. As explication, briefly consider the logic of this theorem. Starting from $x' = f(t, x, \mu)$ we take the partial derivative with respect to μ. Using the chain rule, and rearranging the order of differentiation, leads to the expression $y' = f_x(t, \varphi(t, \tau, \xi, \mu), \mu)y + f_\mu(t, \varphi(t, \tau, \xi, \mu), \mu)$ with solution $y = \frac{\partial \varphi}{\partial \mu}$. Hence we have a differential system and, referencing previous theorems, one which satisfies the conditions necessary for continuous solutions. Yet this is not quite as strong a statement as we would like. In order for Newton's method to converge, we need the second partial derivatives to be continuous. Fortunately, with the imposition of further conditions, this theorem can be extended to provide continuous second partial derivatives.

Corollary F2: *If f_x and f_μ have continuous partial derivative with respect to μ (i.e. $f_{x\,x}$, $f_{x\,\mu}$ and $f_{\mu\,\mu}$ are continuous), then $\frac{\partial^2 \varphi}{\partial \mu^2}$ is continuous.*

We work from the same circumstance as above. If y is our solution $\frac{\partial \varphi}{\partial \mu}$, then in order to obtain $\frac{\partial^2 \varphi}{\partial \mu^2}$, we need to take the partial μ derivative of y. Taking this derivative of our equation $y' = f_x(t, \varphi(t, \tau, \xi, \mu), \mu)y + f_\mu(t, \varphi(t, \tau, \xi, \mu), \mu)$ and letting $z = \frac{\partial^2 \varphi}{\partial \mu^2}$ results in the new system,

$z' = f_x(t, \varphi(t, \tau, \xi, \mu), \mu)z + f_{x\,\mu}(t, \varphi(t, \tau, \xi, \mu), \mu)\, y\,(y + 1) + f_\mu(t, \varphi(t, \tau, \xi, \mu), \mu)\, y + f_{\mu\,\mu}(t, \varphi(t, \tau, \xi, \mu), \mu).$

Note that we have already solved for the first partial derivative y, which was in fact continuous, hence its inclusion above is not of concern. Furthermore, by assumption in the theorem, f_x and f_μ are continuous and by corollary assumption $f_{x\,\mu}$ and $f_{\mu\,\mu}$ are as well. Therefore, again we have a differential system with continuous solution $z = \frac{\partial^2 \varphi}{\partial \mu^2}$.

So we have established the conditions necessary for continuity with respect to parameters and continuity of the second partial derivatives. These both will be of use later in the discussion of Newton's Method. For now, we temporarily end our treatment of the necessary theory and consider its use with respect to our model.

Section III

Application: Simple Pendulum with Forcing

Thus far we have established some of the basic theory necessary to provide for a solution to our more complicated system of differentiable equations and also discussed some theory that will be of future use. We can now consider the simple pendulum with forcing through the use of numerical solutions. Note that as we have shown that the behavior of the single mass distribution pendulum is equivalent to the simple single pendulum, observations made here in fact apply to the single forced mass distribution pendulum as well. Energy conservation, choice of forcing function, and some sample behaviors will be discussed.

Energy Conservation

First, we return to the theme of energy conservation. Recall that we would like a system in which there is a possibility for energy gain, though past models have not had this characteristic. So here we will consider dH/dt for a forced pendulum without friction and with generic forcing function *h(t)*. The system of equations we start with is:

$$\frac{d\theta}{dt} = \dot{\theta} \tag{1.1}$$

$$\frac{d\dot{\theta}}{dt} = -\frac{g}{l}\sin\theta + h(t). \tag{1.2}$$

Now we can use the chain rule and say:

$$\frac{dH}{dt} = \frac{\partial H}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial H}{\partial \dot{\theta}}\frac{d\dot{\theta}}{dt}. \tag{1.3}$$

Using our equations of motion above and our expression for *H*, namely

$$H(\theta,\dot{\theta}) = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos(\theta) \tag{1.4}$$

we can state that

$$\frac{dH}{dt} = \frac{\partial H}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial H}{\partial \dot{\theta}}\frac{d\dot{\theta}}{dt} = l^2 m(\dot{\theta}(t))^2 h(t). \tag{1.5}$$

Is it possible that this expression might be positive? While perhaps at this point we cannot authoritatively state that it must be, we can support the claim that it might be. As

long as the angular velocity and the forcing function have the same sign at particular times it is possible for us to have a positive change in energy.

Results: Specific Forcing Function

With that encouragement, we might consider a particular set of values for our constants and a possible forcing function in order to get a feel for the nature of the solutions of the system and the types of behaviors we might expect. Furthermore, in doing this we can more concretely evaluate the possibility of energy gain.

First let's pick our parameter values and forcing function. Let $h(t) = A\cos(wt)$, g = 1, l = 1, A = .1. Thus our equations of motion become:

$$\frac{d\theta}{dt} = \dot{\theta} \tag{2.1}$$

$$\frac{d\dot{\theta}}{dt} = -\sin\theta + .1\cos wt\,. \tag{2.2}$$

We have left w and initial conditions θ_0 and $\dot{\theta}_0$ open for assignment to determine specific behaviors. We will pick two sets of initial conditions and two choices for w values and compare and contrast the corresponding behaviors.

First let's take $\theta_0 = \pi$, $\dot{\theta}_0 = .1$, and $w = 2$. Physically, this represents a gymnast starting in a handstand on the bar with a small amount of initial angular velocity and under a forcing function with period π. Using ODE architect, we observe the following behavior of θ over time:

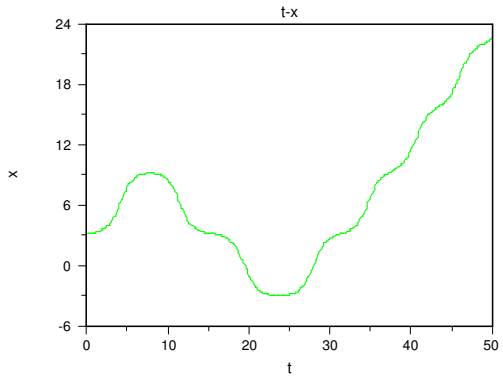


Figure 2.1

So our gymnast starts off by going around the bar in a counterclockwise direction, then goes around the bar clockwise, and once again switches direction, going around the bar

counterclockwise numerous times. With this initial example, let's consider what happens if the forcing function has a shorter period.

Second, let $\theta_0 = \pi$, $\dot{\theta}_0 = 0.1$, and $w = 3$. This corresponds to the same starting position for the gymnast but now under a forcing function with period $2\pi/3$. Using ODE architect, we observe the following behavior of the angle over time:

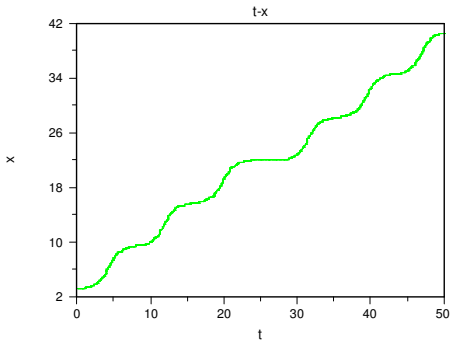


Figure 2.2

So it seems that by increasing the frequency of the forcing our gymnast now only travels around the bar in a counterclockwise manner, where before there were direction changes on the same time interval. In fact, increasing the frequency of the forcing also made the forcing frequency closer to the natural frequency of the solution.

Next, take $\theta_0 = 3.1$, $\dot{\theta}_0 = 0.1$, and $w = 2$. Now our gymnast starts just slightly short of a handstand on the bar with a small about of initial angular velocity and under a forcing function with period π. Using ODE architect, we observe the following behavior:

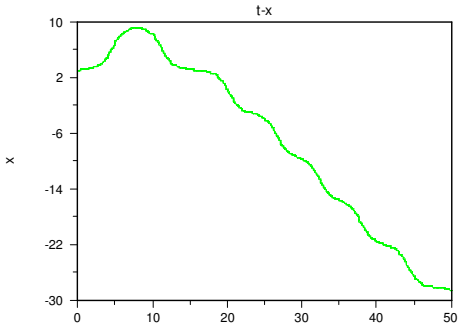


Figure 2.3

Again, we do see the direction change that was present with our $w = 2$ condition before. But now the gymnast only changes direction once and ends up traveling clockwise around the bar a number of times. Interestingly, our θ_0 value has changed only very slightly. Yet recall that our θ_0 value is around π. Physically it makes sense that a small

deviation from the handstand "balance point" might lead to qualitatively different behavior.

Finally, let $\theta_0 = 3.1$, $\dot{\theta}_0 = 0.1$, and $w = 3$. This corresponds to the same starting position for the gymnast but now our forcing function again has period $2\pi/3$. Over time the angle changes as,

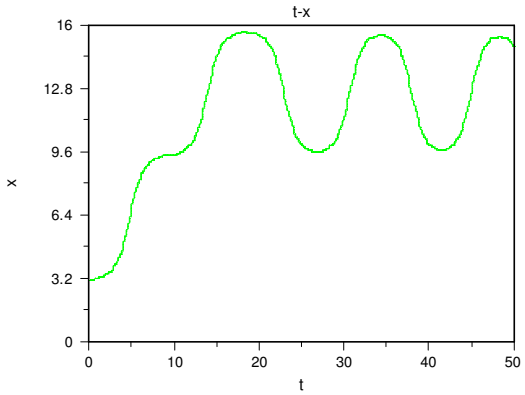


Figure 2.4

This behavior appears qualitatively different than our other solutions. After a few successfully counterclockwise swings, the gymnast ends up oscillating back and forth, not quite making it over the bar each time. So in this case, increasing the forcing frequency has not helped the gymnast keep going around the bar in one direction as it did in the case of the first set of initial conditions. From this variety of behaviors, it is clear that the behavior of solutions to this system is rather complicated and has an intricate dependence on our choice of initial conditions and parameter values.

Double Pendulum with Mass Distribution

With simple pendulum models and a manner of energy increase established, let's consider the addition of a hinge point to our pendulum. While mathematically there is not a strong motivation for this complication, physically it is a central aspect of giant swings. Therefore, as is the case with friction, it is important that we include it. Furthermore, with the tools we have already developed, acquiring equations of motion is a reasonable task and this model leads nicely into our final model.

Equations of Motion

First, let's consider the physical setup and define the necessary parameters.

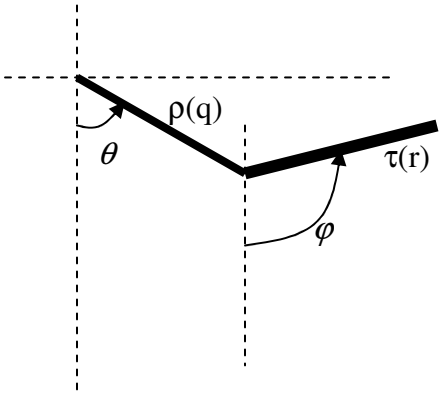


Figure 3.1

Variables:

θ = angle between rod one and vertically downward

$\dot{\theta}$ = angular velocity of first angle

φ = angle between rod two and vertically downward

$\dot{\varphi}$ = angular velocity of second angle

Parameters:

g = gravity

b = coefficient of friction

l = length of rod one

j = length of rod two

q = position along rod one

r = position along rod two

ρ(q) = density function rod one

τ(r) = density function rod two

B = moment zero of rod one = $\int_0^l \rho(q)dq$

C = moment one of rod one = $\int_0^l q\rho(q)dq$

D = moment two of rod one = $\int_0^l q^2\rho(q)dq$

H = moment zero of rod two = $\int_0^j \tau(r)dr$

K = moment one of rod two = $\int_0^j r\tau(r)dr$

S = moment two of rod two = $\int_0^j r^2\tau(r)dr$

Defined as such we have energy expressions,

$$T(\theta,\dot{\theta},\varphi,\dot{\varphi}) = \frac{D}{2}\dot{\theta}^2 + \frac{1}{2}(Hl^2\dot{\theta}^2 + S\dot{\varphi}^2 + 2Kl\dot{\varphi}\dot{\theta}\cos\varphi) \tag{3.1}$$

$$U(\theta,\dot{\theta},\varphi,\dot{\varphi}) = -Cg\cos\theta - g(Hl\cos\theta + K\cos\varphi) \tag{3.2}$$

Using the Lagrangian to calculate the equations of motion and combining in a frictional term as in previous models we have:

$$\frac{d\theta}{dt} = \dot{\theta} \tag{3.3}$$

$$\frac{d\theta}{dt} = \frac{-g(C+Hl)\sin\theta + Kl\dot{\varphi}^2\sin\varphi - Kl\ddot{\varphi}\cos\varphi}{D + Hl^2 + S} - b\dot{\theta} \tag{3.4}$$

$$\frac{d\varphi}{dt} = \dot{\varphi} \tag{3.5}$$

$$\frac{d\dot{\varphi}}{dt} = \frac{-K(g\sin\varphi + l\ddot{\theta}\cos\varphi)}{S} \tag{3.6}$$

Clearly, this is a complex model of four equations. We might initially hope to relate it to our point mass pendulum—recall that in the case of the single pendulum a mass distribution pendulum corresponded to a particular point mass pendulum. Is this also true for the double pendulum, or is the move to a mass distribution a necessary complication?

An Equivalent Simple Double Pendulum?

In fact, what we are asking ourselves here is whether we can find appropriate arm lengths and bob masses to equate our double mass distribution pendulum to a double pendulum. This is best considered through studying the respective energies. We will find that this is not possible.

In approaching this question, let's first consider a simple case. Suppose we have a double pendulum with no mass on the first rod and two point masses m_1 and m_2 located

at distances l_1 and l_2 from the hinge point, with total length of rod one given by l. That is, the pendulum,

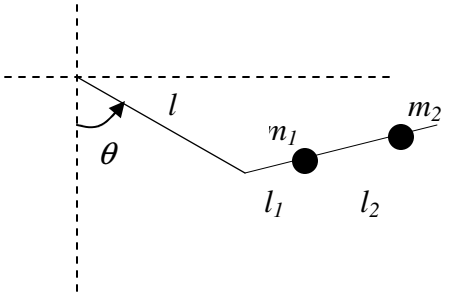


Figure 4.1

Consider the energies of a double pendulum and our special case double mass distribution pendulum. Can we create a correspondence between the constants in these two expressions as before so as to show the two models are equivalent? The special case double mass distribution pendulum energies are given by,

$$T(\theta,\dot{\theta},\varphi,\dot{\varphi}) = \frac{1}{2}(l^2(m_1+m_2)\dot{\theta}^2 + (l_1^2 m_1 + l_2^2 m_2)\dot{\varphi}^2 + 2\dot{\varphi}\dot{\theta}l(l_1 m_1 + l_2 m_2)\cos(\theta-\varphi) \quad (4.1)$$

$$U(\theta,\dot{\theta},\varphi,\dot{\varphi}) = -g(l(m_1+m_2)\cos\theta + (l_1 m_1 + l_2 m_2)\cos\varphi)). \quad (4.2)$$

The double point mass pendulum with point masses has energies,

$$T(\theta,\dot{\theta},\varphi,\dot{\varphi}) = \frac{1}{2}(\bar{m}_1\bar{l}_1^{\,2} + \bar{l}^{\,2}\bar{m}_2)\dot{\theta}^2 + \bar{m}_2\bar{l}_2^{\,2}\dot{\varphi}^2 + 2\bar{m}_2\bar{l}_2\cos(\theta-\varphi) \quad (4.3)$$

$$U(\theta,\dot{\theta},\varphi,\dot{\varphi}) = -g((\bar{l}\bar{m}_2 + \bar{l}_1\bar{m}_1)\cos\theta + \bar{l}_2\bar{m}_2\cos\varphi). \quad (4.4)$$

Matching up corresponding constant coefficients yields the equations,

$$l(m_1+m_2) = \bar{l}\bar{m}_2 + \bar{m}_1\bar{l}_1 \quad (4.5)$$

$$l_1 m_1 + l_2 m_2 = \bar{l}_2\bar{m}_2 \quad (4.6)$$

$$l^2(m_1+m_2) = \bar{l}^{\,2}\bar{m}_2 + \bar{m}_1\bar{l}_1^{\,2} \quad (4.7)$$

$$l_1^2 m_1 + l_2^2 m_2 = \bar{m}_2\bar{l}_2^{\,2} \quad (4.8)$$

$$l(l_1 m_1 + l_2 m_2) = \bar{m}_2\bar{l}_2 \quad (4.9)$$

Let's consider equations (4.8) and (4.9). If we multiply equation (4.9) by $\bar{l}_2$ we should end up with equation (4.8). Yet doing this, we see that this implies

$$\begin{aligned} \bar{l}_2 l &= l_1 \\ \bar{l}_2 l &= l_2 \end{aligned} \tag{4.10}$$

which in turn indicates that l_1 and l_2 must be equal. Yet this is not a constraint we can put on our pendulum, for l_1 and l_2 are the lengths of the two arms of our simple case mass distribution pendulum—lengths that are predetermined and need not be equal. Hence by considering a simple case, we see that the double mass distribution pendulum need not correspond to a double point mass pendulum. The behaviors of the two models are qualitatively different, thus the complication in taking the double point mass pendulum to a double mass distribution pendulum is a necessary one.

Check Our Work

Before relying on these equations of motion to build our final model of the swinging gymnast, we ought to in some way verify our results thus far. In order to do this, we first make a convenient change of variables that will allow for an efficient method of checking our equations.

Let's now choose our second variable to be ψ. This will not be the angle as measured from vertical (i.e. φ), but the angle created between the first and second arm of the pendulum. Physically, this variable more accurately reflects the processes we are concerned with. Furthermore, framing our model as such will be useful once we bring in forcing. This change in variables is done by the replacements

$$\varphi = \pi + \theta - \psi \tag{5.1}$$

$$\frac{d\varphi}{dt} = \frac{d\theta}{dt} - \frac{d\psi}{dt} \tag{5.2}$$

in the energies of the double mass distribution pendulum before the use of Lagrange's equation. This results in the system of equations:

$$\frac{d\theta}{dt} = \dot{\theta} \tag{5.3}$$

$$\frac{d\dot{\theta}}{dt} = \frac{-g(C+Hl)\sin\theta + gK\sin(\theta-\psi) + S\ddot{\psi} + Kl(\dot{\psi}(\dot{\psi}-2\dot{\theta})\sin\dot{\psi} - \ddot{\psi}\cos\psi)}{D + Hl^2 + S - 2KlCos(\psi)} \tag{5.4}$$

$$\frac{d\psi}{dt} = \dot{\psi} \tag{5.5}$$

$$\frac{d\dot{\psi}}{dt} = \frac{-gK\sin(\theta-\psi) + Kl\dot{\theta}^2\sin\psi + \ddot{\theta}(S - Kl\cos\psi)}{S} \tag{5.6}$$

With this reworking of our system of equations we are now ready to check our work. Suppose we set $\psi = \pi$ and $\dot{\psi} = 0$. Physically, we are fixing the angle between the two rods to always be π. Doing this, we would expect our mass distribution double pendulum to behave as a mass distribution single pendulum. Does this in fact occur?

First, we need to do a bit of work to construct our expected answer. Given a single pendulum with one density function for the upper portion and another for the bottom portion what would we expect our equations of motion to be (density functions and constants as defined previously)? Setting up the energies and once again using Lagrangian mechanics, we end up with an expected equation of motion

$$\ddot{\theta} = \frac{-g(C + K + Hl)\sin\theta}{D + 2Kl + Hl^2 + S}. \tag{5.7}$$

With this information at hand, we can consider what happens in our double mass distribution pendulum system of equations when we fix $\psi = \pi$, $\dot{\psi} = 0$, $\ddot{\psi} = 0$. Ideally, we would end up with the result above. In fact after many simplifications we get this result. Thus happily is seems that our system of equations does pass the test.

Section IV

Having developed the essential mathematical tools and achieved an intuitive understanding of giant swings through considering particular pendulum models, we can now build our final model of the swinging gymnast. After mapping out the physical model, we establish the equations of motion building from the pendulum models already discussed. Our unique use of forcing is useful from both a mathematical and physical perspective. Energy conservation will be again addressed. With a reasonable model at hand, we can attempt to find a solution that achieves a certain behavior. Here we draw on our previous observations on the continuity of solutions with respect to the parameters in order to justify the use of Newton's method.

Physical Set Up

From previous investigations, recall that perhaps the best model we can put forth is a double mass distribution pendulum affected by friction and some sort of forcing. Recalling the parameters of our previous double mass distribution model, we'll add only a coefficient of friction for now, call it *b*. Note that we are continuing to work from the system in terms of ψ and θ. We have the system:

$$\frac{d\theta}{dt} = \dot{\theta} \tag{1.1}$$

$$\frac{d\dot{\theta}}{dt} = \frac{-g(C+Hl)\sin\theta + gK\sin(\theta-\psi) + S\ddot{\psi} + Kl(\dot{\psi}(\dot{\psi}-2\dot{\theta})\sin\psi - \ddot{\psi}\cos\psi)}{D+Hl^2+S-2KlCos(\psi)} - b\dot{\theta} \tag{1.2}$$

$$\frac{d\psi}{dt} = \dot{\psi} \tag{1.3}$$

$$\frac{d\dot{\psi}}{dt} = \frac{-gK\sin(\theta-\psi) + Kl\dot{\theta}^2\sin\psi + \ddot{\theta}(S-Kl\cos\psi)}{S} \tag{1.4}$$

This is a fairly complicated system of four equations with considerable interdependence and many nonlinear terms. It is in fact intricate enough that we would like to be able to do something to make it even slightly simpler. A wise use of forcing might just achieve this for us.

Forcing

Recall that our second variable is now the angle between the first and second arm of the pendulum. In fact, in attempting giant swings, gymnasts use muscle action to control this angle. Thus it seems logical that instead of tacking a forcing function onto our equations as is, we might instead choose to define the angle between the two arms of the pendulum. That is, let's set

$$\psi = \pi - \varepsilon \sin \omega t. \tag{2.1}$$

Doing the appropriate differentiation and substitution into the first two equations of our system, we get:

$$\frac{d\theta}{dt} = \dot{\theta} \tag{2.2}$$

$$\frac{d\dot{\theta}}{dt} = \frac{-AS\omega^2 \sin(\omega t) + gK\sin(A\sin(\omega t) - \theta) - g(C + Hl)\sin(\theta) - AKl\omega(\omega\cos(\omega t)\cos(A\cos(\omega t)) + \sin(\omega t)\sin(A\cos(\omega t))(A\omega\sin(\omega t) + 2\dot{\theta}))}{D + Hl^2 + S - 2Kl\cos(A\cos(\omega t))} - b\dot{\theta} \tag{2.3}$$

Though still far from straightforward, this system of equations appears slightly more manageable. We have less than our previous four equations, though we have introduced a time dependence, so really if we were to write the system autonomously we would have three equations, namely:

$$\frac{d\theta}{dt} = \dot{\theta} \tag{2.4}$$

$$\frac{d\dot{\theta}}{dt} = \frac{-AS\omega^2 \sin(\chi) + gK\sin(A\sin(\chi) - \theta) - g(C + Hl)\sin(\theta) - AKl\omega(\omega\cos(\chi)\cos(A\cos(\chi)) + \sin(\chi)\sin(A\cos(\chi))(A\omega\sin(\chi) + 2\dot{\theta}))}{D + Hl^2 + S - 2Kl\cos(A\cos(\chi))} - b\dot{\theta} \tag{2.5}$$

$$\frac{d\chi}{dt} = \omega \tag{2.6}$$

where $\chi = \omega t$. Still, it seems that there may be mathematical benefits to the employment of forcing in this slightly strange manner.

But before we go much further, we ought to justify our choice of forcing function. This forcing sets the values of ψ, which, recall, is the angle between the torso and legs of the gymnast, to be between $\pi - \varepsilon$ and $\pi + \varepsilon$. In fact, provided epsilon is small enough, perhaps at most $\pi/2$, this is quite physically reasonable. Secondly, note that a periodic

function has been chosen. This is logical in light of the nature of the technique taught for giant swings. Additionally, at $t = 0$, we would like our gymnast to be able to start in a handstand on top of the bar. Therefore, the desired initial angle between the torso and legs of the gymnast ought to be π, as this choice of forcing function provides for. Thus it seems that our choice of forcing function is sufficiently realistic.

Examples

In order to get a feel for this system, let's briefly consider its behavior under particular initial condition and parameter values. While certainly this is a miniscule sample of the possible behaviors of the system we ought to have some provisional understanding to start from.

Let's consider two behaviors. Our first set of parameters will be as follows: $\varepsilon = .75$, $\omega = 3$, $b = .1$, $S = g = K = C = H = l = D = 1$. We'll take an initial condition $\theta = 3.14$, $\dot{\theta} = -.1$. Our resulting solution on a fairly long term scale is as follows:

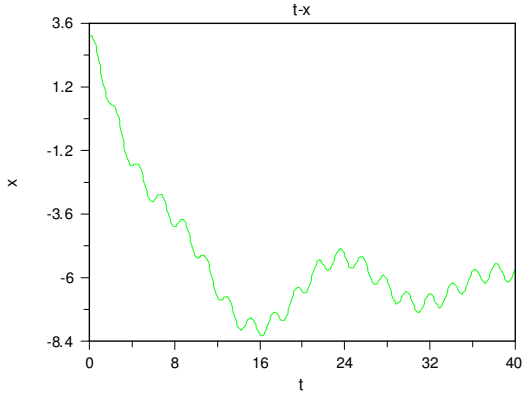


Figure 3.1

What is going on here? Our gymnast first goes around the bar counterclockwise a few times, then goes back and forth without making it all the way around the bar. Note also the "bumpiness" of the graph indicates small numerous direction changes throughout the swing. This would be very odd behavior for a swinging gymnast.

Our second set of parameters is similar to the first. Except now we reduce the coefficient of friction, taking $b = 0.01$. We also change our forcing parameters, setting $\varepsilon = -1.4$ and $\omega = 1$. Our initial condition is $\theta = 3.14$, $\dot{\theta} = 0.1$, that is, we have changed the direction of the initial angular velocity. The resulting solution is plotted below.

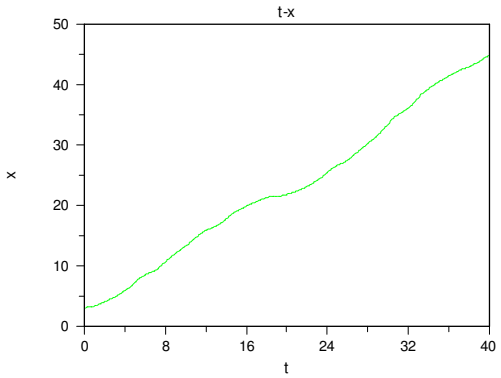


Figure 3.2

Here, our gymnast appears to behave as we would like her to. As time goes on, she continually goes around the bar counterclockwise in a fairly smooth manner—the bumpy texture noted on the previous graph is gone. Again, while these two examples are merely a small sample, through considering them we have reached a starting point for further study. Additionally, we should note that in observing the behaviors of these solutions, we again find the signs of chaos that were found in the single forced pendulum system.

Energy Conservation

With these examples at hand, once again we recall the perennial question of energy conservation. From our previous work with forcing and friction we would expect that it would be possible to gain energy using this model. Note that while Lagrangian mechanics does guarantee conservation of energy, in fact our modifications through the inclusion of forcing are what alter our situation. While we put off a complete analysis until specific parameter values have been chosen, we can again use the chain rule and our equations of motion to show that the change in energy expression is not identically zero. Once again, it seems plausible that particular choices of parameter values would allow for a positive expression. With this encouragement, or at least lack of discouragement, and our previous results we move on, perhaps to return to this issue under the consideration of a particular solution.

Defining Success

With a model to work from, we can discuss in more detail the desired results. Our first definition of success will be as follows. Take a gymnast starting in handstand on the bar with minimal angular velocity. At the end of one period of the forcing function, we

would like the gymnast to have returned to a handstand having completed at least one revolution around the bar. We would also like the angular velocity of the gymnast to be a small positive number. Mathematically, we can say we would like to satisfy the conditions

$$\begin{aligned} &\theta_0 = -\pi \\ &\theta_0^{'} = \beta \qquad where\ 0 < \beta < small \\ &t_1 = \left|2\pi / \omega\right| \end{aligned} \tag{5.1}$$

$$\begin{aligned} &\theta_{t_1} = \pi \\ &\theta_{t_1}^{'} = \chi \qquad where\ 0 < \chi < small \end{aligned} \tag{5.2}$$

The question now becomes, how ought we go about satisfying these conditions? Though the initial condition and end condition are predetermined, we have a number of parameters to work with. Of these parameters, those in the forcing function are perhaps the best to consider, as the other parameters are fairly physically pre-defined (we can hardly hope to dictate the force of gravity or the moments of the gymnast's torso). So, with the other parameters and an initial condition defined, we would like to determine values for the forcing parameters A and ω such that the end condition is satisfied. While initial estimations of A and ω can get us in the vicinity of an answer to this question, it is best to go about this in a more structured manner. An adaptation of Newton's method is of use in this instance.

Newton's Method

Newton's method is a good tool for interpolation as it is fairly straightforward yet in most cases will converge rapidly to a solution. Before considering our specific employment of Newton's method we ought to develop the concepts behind it and consider the simple cases. First we take on the one dimensional case, then expand into two dimensions and parameters. Finally, we consider Newton's method as applied to the parameters in our multidimensional system, where the endpoint of evaluation depends on the parameters. For further discussion, see Heath and Burden/Faires.

One Dimensional Case

Suppose we have a nonlinear equation, call it *f(x)*, that we would like to find the zeros of. That is, we would like to find the x values such that *f(x)* = 0. First we consider the truncated Taylor Series of *f(x)*:

$$f(x+h) \approx f(x) + f'(x)h. \tag{6.1}$$

Here, we have approximated the function using its tangent line at the point *x*. Can we find the zeros here? Setting the left hand side equal to zero we see that

$$h = -f(x)/f'(x). \tag{6.2}$$

Therefore, in order to have *f(x)* = 0 we ought to take

$$x_{new} = x - f(x)/f'(x). \tag{6.3}$$

Let's formalize this into a cleaner algorithm. First take an *x* value, call it x_0, that is relatively near a zero of *f(x)*. Starting from this initial guess, we can find an *x* value that is closer to being a zero using

$$x_{k+1} = x_k - f(x_k)/f'(x_k). \tag{6.4}$$

By continuing to iterate with (6.4) we quickly close in on the zero of the function *f(x)*. In fact, if our initial guess is reasonable and the zero we're attempting to find is a single root, then Newton's method should converge quadratically to the zero of the function. In other words, the number of correct digits should double with each iteration. In this respect, we ought to consider the convergence theorem,

Theorem F2: *Let* $f \in C^2$ *on an interval [a, b]. If there exists a zero of the function, call it p, such that p is in the interval [a, b], f(p) = 0, and* $f'(p) \neq 0$ *then there exists a* $\delta > 0$ *such that Newton's method generates a sequence* $\{p_n\}_{n=1}^{\infty}$ *that converges to p for an initial approximation* p_0 *within* $[p - \delta, p + \delta]$.

The $f'(p) \neq 0$ condition guarantees that the zero of our function is not tangent to the x-axis. Note that there are few limitations on δ; it may be that we need to be within .00001 of our solution in order to converge to it. Before considering the proof of this theorem, let's note the fixed point theorem which will be necessary to our proof. We have,

Theorem F3 (Fixed Point Theorem): *Let g be continuous on an interval [a, b] and suppose* $g(x) \in [a,b]$ *for all x in [a, b]. Furthermore, let g'(x) exist on (a, b) with*

$|g'(x)| \le k < 1$ *for all x in (a, b). The if* p_0 *is a value within [a, b] then the sequence* $p_n = g(p_{n-1})$ *(for* $n \ge 1$*) converges to the unique fixed point p in [a, b].*

With the fixed point theorem in mind, let's consider the proof of our convergence of Newton's method theorem. As we would like to invoke the fixed point theorem, we should attempt to use our assumptions to satisfy the conditions of the fixed point theorem. First, we set up the iterative equations for Newton's method. Recall our process can be written as

$$g(x) = x - f(x)/f'(x). \tag{6.5}$$

To satisfy the conditions of the fixed point theorem, we need to find a k in (0, 1) and an interval [p - δ, p + δ], such that g map this interval onto itself and $|g'(x)| \le k < 1$ for all x in this interval.

By our assumptions, *f'(p)* is nonzero and *f'* is continuous. Therefore there must exist a $\delta_1 > 0$ such that *f'(x)* is nonzero for x in $[p - \delta_1, p + \delta_1]$. Therefore our representation *g(x)* is both defined and continuous on this interval. Furthermore, for x in this interval we can write,

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{f'(x)f'(x)} = \frac{f(x)f''(x)}{[f'(x)]^2}. \tag{6.6}$$

Since f is twice differentiable continuous, g' must be continuous on this interval. As we assume $f(p) = 0$, we can say,

$$g'(p) = \frac{f(p)f''(p)}{[f'(p)]^2} = 0 \tag{6.7}$$

Since g' is continuous, this implies that for any $0 < k < 1$ there exists a δ (where $0 < \delta < \delta_1$) such that $|g'(x)| \le k$ for any x in the interval [p – δ, p + δ]. With the initial portion of the fixed point theorem established, we must now show that g maps this interval onto itself. Using the mean value theorem, we can state that for some τ between x and p, we have

$$|g(x) - g(p)| = |g'(\tau)||x - p|. \tag{6.8}$$

Thus,

$$|g(x) - p| = |g(x) - g(p)| = |g'(\tau)||x - p| \le k|x - p| < |x - p|. \tag{6.9}$$

Because x is in our interval [p – δ, p + δ] we know that $|x - p| < \delta$ which implies that $|g(x) - p| < \delta$. Or, in other words, g must map our interval onto itself. At this point, we have satisfied the conditions of the fixed point theorem, and thus can conclude that Newton's method must converge to our solution p, for any p_0 in the interval [p – δ, p + δ]. With this basic understanding of Newton's method, let's begin to consider how it can be adapted and applied to more complicated systems.

Two Dimensions

As the system we would ultimately like to apply Newton's method to is multidimensional, we ought to briefly consider the manner in which Newton's method can be expanded into two dimensions. In fact, Newton's method looks very similar here, except we now employ matrixes and vectors. Finding the zeros of the Taylor approximation through solving for h, as above, leads to the result:

$$x_{k+1} = x_k - J_f^{-1}(x_k) f(x_k). \tag{7.1}$$

Here f is a vector of two equations, x is a vector of our two independent variables, and J^{-1}_f is the inverse of the Jacobian matrix. That is,

$$J_f = \begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{pmatrix}. \tag{7.2}$$

So even in expanding to higher dimensions, Newton's method maintains much of its simplicity. Can we now evolve Newton's method so as to apply it to parameter interpolation?

Newton's Method for Parameters

Now the extra efforts extended in the theory section begin to pay off. As we have established continuity of our solutions with respect to parameters, we can now label our parameters as variables. That is, instead of searching for an x vector that zeros the function, we seek a parameters vector that zeros the function while keeping the x values constant. If our parameter vector is given by

$$P = \begin{pmatrix} a \\ b \end{pmatrix} \tag{7.3}$$

then now we have a function $f(P, x)$ that we want to find the zeros of by altering P, that is x remains fixed. Taking an initial guess P_0, we then use

$$P_{k+1} = P_k - J_f^{-1}(P_k) f(P_k) \tag{7.4}$$

to reach a better approximation of the parameter values necessary to zero the function. Note that here

$$J_f = \begin{pmatrix} \frac{\partial f_1}{\partial a} & \frac{\partial f_1}{\partial b} \\ \frac{\partial f_2}{\partial a} & \frac{\partial f_2}{\partial b} \end{pmatrix}. \tag{7.5}$$

So we have expanded Newton's method so that it can be applied to a multidimensional system to interpolate the parameter values necessary to zero a function. While this is significant progress, there is still work to be done before we can apply Newton's method to our particular problem.

An Interesting Dependence

Unfortunately, our system has an additional complication that must be addressed. As we change the parameter vector, the time endpoint we consider also moves. That is x has dependence on the parameter vector's relationship with time. Thus we must apply the chain rule when finding the Taylor Series of the function we seek the zeros of. This only affects our Jacobian matrix of the Newton's method process. We now have,

$$J_f = \begin{pmatrix} \frac{\partial f_1}{\partial t}\frac{dt}{da} + \frac{\partial f}{\partial a} & \frac{\partial f_1}{\partial t}\frac{dt}{db} + \frac{\partial f}{\partial b} \\ \frac{\partial f_2}{\partial t}\frac{dt}{da} + \frac{\partial f}{\partial a} & \frac{\partial f_2}{\partial t}\frac{dt}{db} + \frac{\partial f}{\partial b} \end{pmatrix}. \tag{8.1}$$

Clearly, this additional dependence requires a substantial amount of differentiation. Fortunately, besides this change the process of Newton's method is otherwise maintained. Furthermore, when we consider the dependence particular to our problem, we find that many of these derivatives are zero. This simplifies our calculations significantly. Let's now apply this result.

Section V

With the theory of Newton's method established we can now consider its application to our particular problem. As always, we begin with a more basic system by applying Newton's method to the simple forced pendulum with friction model. In this application we encounter a few difficulties that require further reflection. After examining the possible causes of these troubles we develop mechanisms to help us once again modify Newton's method so that it can be useful in finding solutions. Finally, we attempt to consider the application of Newton's method to our more complicated model.

Application of Newton's Method

Having adapted Newton's Method so it is suited to our particular problem, we can now consider its application. Before approaching our rather complicated final model, we will work with the simple forced pendulum model. So we begin with the system,

$$\begin{aligned} \frac{d\theta}{dt} &= \dot{\theta} \\ \frac{d\dot{\theta}}{dt} &= -\sin\theta + g(t) - b\dot{\theta} \end{aligned} \tag{1.1}$$

where *g(t)* is our forcing function, and *b* is the coefficient of friction. To simplify the equations we have designated some parameter values, namely $g = l = 1$.

Note first that we ought to confirm that we satisfy the conditions necessary for the use of Newton's method. The system must be continuous with respect to the parameters and have continuous second partial derivatives. Referring to the theory section, we see that these statements hold true if the right hand side of the system of equations is continuous, Lipschitz, and must have continuous second partial derivatives with respect to the parameters. Clearly (1.1) above is constructed of analytic functions and so it is continuous and infinitely differentiable with respect to θ, $\dot{\theta}$, t, and our parameters—provided our choice of *g(t)* is a similarly well behaved function. As noted previously, differentiability implies Lipschitz. Thus, provided our choice of *g(t)* is a reasonable analytic function, our model satisfies the necessary conditions for the convergence of Newton's method.

With that point established, let's begin with the forcing function $g(t) = A(1-\cos(\omega t))$. Recall that we would like to find forcing parameters A and ω so that our solution satisfies

$$\begin{aligned} t_1 &= |2\pi/\omega| \\ \theta_{t_1} &= \pi \\ \theta'_{t_1} &= \chi \qquad where\ 0 < \chi < small. \end{aligned} \tag{1.2}$$

We begin with the initial conditions

$$\begin{aligned} \theta_0 &= -\pi \\ \theta'_0 &= \beta \qquad where\ 0 < \beta < small \end{aligned} \tag{1.3}$$

where here we will specify $\chi = \beta = .1$ and pick our remaining parameter values, that is, the coefficient of friction, to be $b = .05$. Furthermore, we take an initial guess for (A, ω) of (.0085813, .719551). This parameter vector gives an endpoint solution of $(\theta, \dot{\theta}) = (1.04855, -1.52033)$. Recall our expanded Jacobian matrix that reflects our additional dependence:

$$J_f = \begin{pmatrix} \frac{\partial f_1}{\partial t}\frac{dt}{da} + \frac{\partial f}{\partial a} & \frac{\partial f_1}{\partial t}\frac{dt}{db} + \frac{\partial f}{\partial b} \\ \frac{\partial f_2}{\partial t}\frac{dt}{da} + \frac{\partial f}{\partial a} & \frac{\partial f_2}{\partial t}\frac{dt}{db} + \frac{\partial f}{\partial b} \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta}{\partial t}\frac{dt}{dA} + \frac{\partial \theta}{\partial A} & \frac{\partial \theta}{\partial t}\frac{dt}{d\omega} + \frac{\partial \theta}{\partial \omega} \\ \frac{\partial \dot{\theta}}{\partial t}\frac{dt}{dA} + \frac{\partial \dot{\theta}}{\partial A} & \frac{\partial \dot{\theta}}{\partial t}\frac{dt}{d\omega} + \frac{\partial \dot{\theta}}{\partial \omega} \end{pmatrix}. \tag{1.4}$$

Yet note that for our particular system these expressions simplify considerably. Thus we have,

$$J_f = \begin{pmatrix} \frac{d\theta}{dA} & \frac{\partial \theta}{\partial t}\frac{dt}{d\omega} + \frac{\partial \theta}{\partial \omega} \\ \frac{d\dot{\theta}}{dA} & \frac{\partial \dot{\theta}}{\partial t}\frac{dt}{d\dot{\theta}} + \frac{\partial \dot{\theta}}{\partial \omega} \end{pmatrix}. \tag{1.5}$$

Additionally, $\frac{\partial \theta}{\partial t}$ is our first equation of motion and $\frac{\partial \dot{\theta}}{\partial t}$ is our second equation of motion. $\frac{dt}{d\omega}$ refers to our endpoint dependence on the parameter ω. Since $t = |2\pi/\omega|$ we have $\frac{dt}{d\omega} = -2\pi/\omega^2$ $(\omega \geq 0)$, $\frac{dt}{d\omega} = 2\pi/\omega^2$ $(\omega < 0)$. The remaining partial derivatives are calculated using the chain rule for our system of equations along with numerical solutions. Using this as our Jacobian matrix and carrying out Newton's

method with *Mathematica* under the rubric outlined in the previous section, the result after the second iteration is the parameter vector, (-.530915, -0.180627), which puts our endpoint solution position as $(\theta, \dot{\theta}) = (-201.6871, -8.1832)$. This ought to seem rather suspicious. Note that we have not really improved our endpoint situation from our starting parameter vector (noted above). In fact, both our θ and $\dot{\theta}$ values are substantially farther away from our desired endpoint result. To get a better idea of the solution this parameter vector describes, let's consider the phase plane. We have,

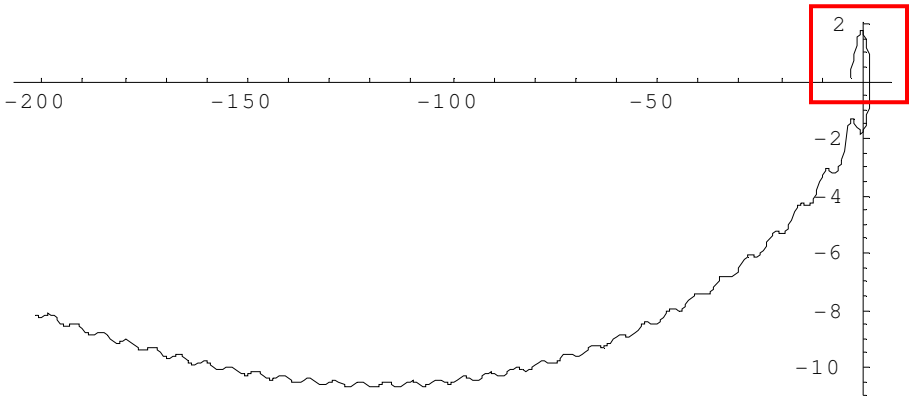


Figure 1.1

Clearly this is far from the solution behavior we desire. It seems that we have overshot our desired parameter values. Recall that the type of phase plane behavior we desire looks something like this,

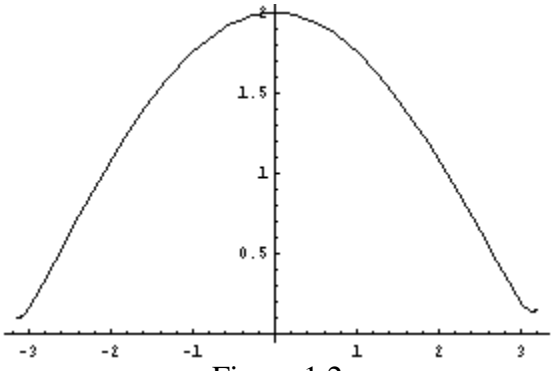


Figure 1.2

where our initial condition is on the left hand side of the plot and our end condition the right hand side. Studying the plot of the solution for our parameter vector, aspects of this behavior are evident. Note the boxed in section of the plot. Locally this behavior seems to resemble our desired result; it appears that we do pass nearby our desired endpoint. Yet we miss our actual desired endpoint and the time at which we are close to the desired value does not correspond to the endpoint time. As we continue to follow the solution for

the remaining time, we quickly escape from the area of the desired solution. Our resulting endpoint solution is nowhere near our desired result. Continuing to iterate Newton's method is unwise; we have left the area of our solution and so we are not starting close enough to yield a result. In fact, another iteration of Newton's method here sends our endpoint result towards infinity.

Fixable?

Why has Newton's method failed us here? Our previous efforts would suggest that it ought to work—we have continuity with respect to parameters, we have continuity of second partial derivatives, we have the correct mechanism for finding a solution, and our starting parameter vector would seem to be in a reasonable vicinity of our desired solution. Yet we must also keep in mind the delicate nature of our situation. We are considering a potentially chaotic system and our desired endpoint is very close to an equilibrium point which adds further sensitivity. These two factors make the application of Newton's method difficult here. What can we do to better our situation?

Once again, consider the components of our equations of motion and their contributions with respect to the energy. The forcing function allows for a potential gain in energy while friction takes energy out of the system. Thus friction is what makes successful giants—which require an increase in energy—difficult. So perhaps reducing the influence of friction, or in other words, making the coefficient of friction closer to zero, will help stabilize our system. Additionally, we could attempt to choose an initial parameter vector that starts even closer to the solution in hopes that this will keep us within the solution area. Alternatively, we could adjust Newton's method to decrease the step sizes taken from one iteration to the next. This is possible through putting a scaling factor, some $0 < s < 1$, in front of our step size in Newton's method. In other words, we now have,

$$P_{k+1} = P_k - sJ_f^{-1}(P_k)f(P_k). \tag{2.1}$$

Hence the adjustment of the parameter vector by the linearized Taylor Series has been scaled down and our parameter vector—and thus our solution—undergoes less of a change from one iteration to the next. We observe that at the least, this slows down the convergence of Newton's method. In fact, there is no guarantee that Newton's method

will still converge if we use a scaling factor. While this is hardly sufficient proof, we find that in its application it does help avoid overshooting our parameter solution and still appears to converge to a solution. Intuitively, it seems reasonable that the use of a scaling factor would not prevent convergence though it would certain slow down the rate of convergence. Yet a further, more thorough, discussion is warranted here. With these possibilities at hand, let's see if we can adjust Newton's method to find a solution for the example above.

Less Friction and a Scaling Factor

First, let's decrease the coefficient of friction and see if we can find a parameter vector solution for that system. We take b = .003 and will also use a scaling factor of s = .07. Beginning with the initial parameter vector (A,ω) = (.0025813, .699551), after twelve iterations we have the parameter solution (A,ω) = (.00242168, .717272) which gives the endpoint result $(\theta,\dot{\theta}) = (3.14159, .1)$. Thus using a smaller coefficient of friction, a scaling factor, and a reasonable initial parameter vector guess we have successfully found a solution through Newton's method. Our phase plane resembles the desired phase plane solution. The solution curves in (θ, t) and $(\dot{\theta},t)$ space are respectively,

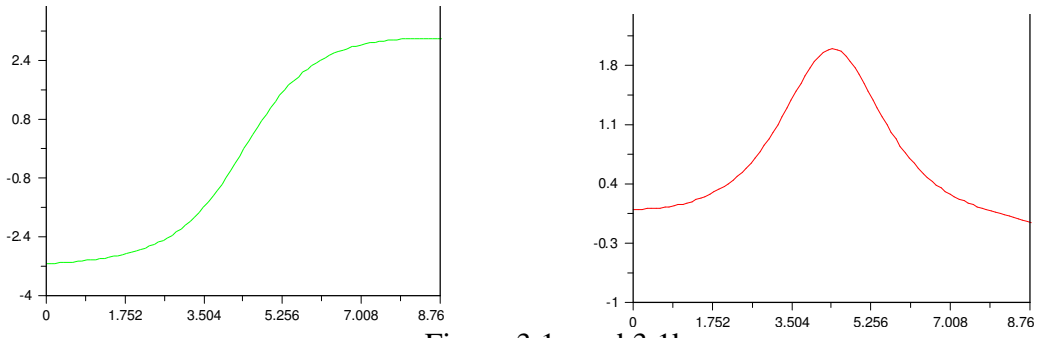


Figure 3.1a and 3.1b

We see in the time plots an initial slow increase in angle, followed by a middle section of high angular velocity which then decreases again as the gymnast reaches the top of the bar. It seems like this sort of forcing "works with" the gymnast's natural swing in a sense. That is, the solution curves are quite smooth and regular looking—nothing particularly dramatic is going on with the forcing, instead it has a rather subtle (but necessary) affect, as might be guessed from the small values of both the amplitude and

the coefficient of friction. In fact, along these lines it is instructive to consider a plot of the change in energy over time (that is, dH/dt). This sheds further light on the nature of the gymnast's swing. For the solution above we have the change in energy plot,

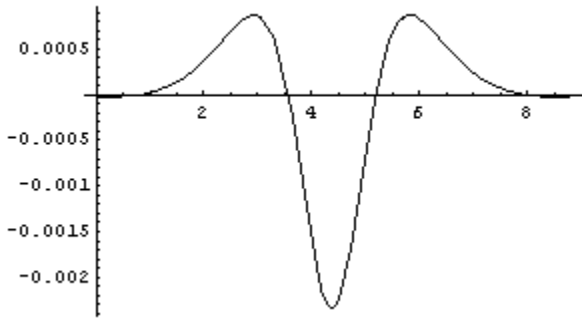


Figure 3.2

We see that initially, there is a very small loss of energy. Yet the net effect quickly becomes positive. Following this phase is an area of significant energy loss at the bottom of the gymnast's swing ($\theta = 0$). Referring to the plots above, this is the point of maximum overall angular velocity for the gymnast—thus it is not surprising that at this point our frictional decelerating term has its largest affect. Next we have another phase of energy increase in the vicinity of $\theta = \pi/2$, symmetric to our previous phase. The gymnast again loses a small amount of energy at the end of her swing. With this understanding of our solution and the success of our modified Newton's method, we return to our system with a larger coefficient of friction and attempt to find a solution.

Increased Friction

Let's approach the $b = .05$ case that gave us so many problems before. But note that we have a solution for the $b = .03$ case which can in fact be of service. That is, these solution values might serve as a reasonable first guess parameter vector for our increased friction system. This is a consequence of the continuity of solutions with respect to parameters—given a solution for a particular parameter value b, we would expect the solution for $b + \delta$ to be similar. First we use our solution parameter vector for b = .03 as an initial parameter vector to get a solution for the $b = .04$ case. After sixteen iterations under a scaling factor of $s = .8$ we have the solution $(A,\omega) = (0.03228, .717233)$. Using this as an initial parameter vector for our $b = .05$ solution with the same scaling factor as before, after ten iterations we have the result $(A,\omega) = (0.0403516, 0.717212)$ which puts us at our desired endpoint (3.1415, .1) for end time 8.76057. Thus a combination of our

adaptations of Newton's method have allowed us to find a solution to the system that originally presented difficulties. Let's again consider a plot of the change in energy. We have,

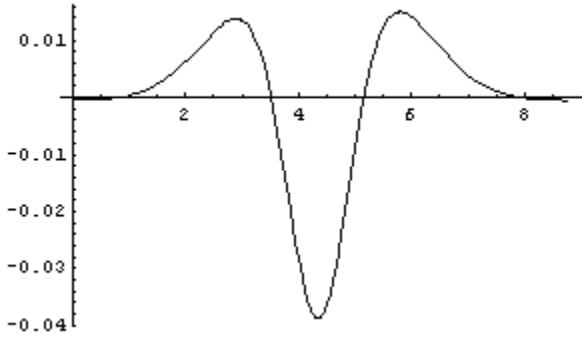


Figure 4.1

Note that this plot is very similar to that of our previous example. It is quite symmetric, with maximum energy loss at the bottom of the gymnast's swing ($\theta = 0$). The magnitudes of the energy loss and energy gain are larger here than above (though still fairly small) yet the plot shapes are virtually identical.

Therefore when Newton's method isn't initially bringing us towards a solution we can try a scaling factor to tone down the steps that we end up taking. Additionally, we can inch our way closer to a solution by starting from a known solution, changing the parameter values that we want to change very slowly, and filling in more solutions to adjust our initial parameter values so we stay in the region of a solution. With these strategies at hand, let's spend some time considering other possible forcing functions for this system.

Similar Forcing

To being with, let's attempt to approach forcing functions much like the one we have a solution for already. We can generalize our previous forcing function slightly and say $g(t) = A(Q\text{-}\cos(\omega t))$, where $Q \in \square$. What happens when we try different Q values in our forcing function? Can we continue to expect to find a solution? In particular, we attempt to take Q to zero, that is, come up with a parameter value solution for $g(t) = -A(\cos(\omega t))$. We use the initial and endpoint condition as in prior models, and once again take coefficient of friction $b = .05$. Our technique here is similar to our methods above. Taking the above $Q = 1$ solution as our initial parameter vector value, we begin

to step down the Q value incrementally, solving those systems for new parameter vectors that are then used as the new initial parameter vector for the next step. Additionally a scaling factor of .7 helps keep us in the region of the solution. Doing this, we find that initially our Q steps can be reasonably sized; $\Delta Q = .1$ to $.2$ works out nicely. Yet once we reach the $Q = .1$ region we encounter some difficulties. Even step sizes of .01 require a substantial scaling factor (roughly $s = .1$). Yet we can reach a solution by using *Mathematica* to create a loop to expediate the process of Newton's method with a scaling factor and stepping down our Q value. Doing so leads us to the result $(A, \omega) =$ (0.492382, 0.480895). Yet this solution is qualitatively different from our previous results. First note the phase plane solution as below,

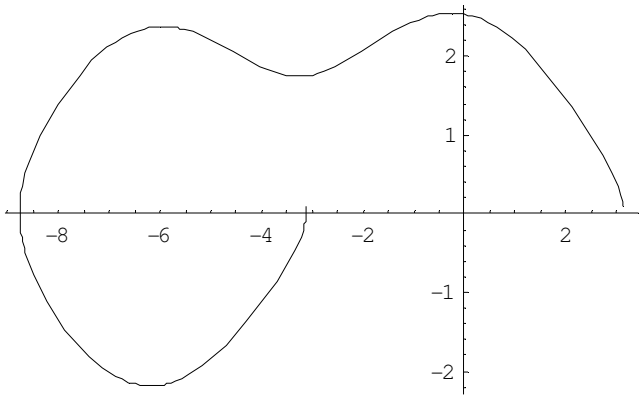


Figure 5.1

Our gymnast here is doing something very different than before. First traveling clockwise around the bar, she then changes direction and travels counterclockwise (the "right" direction) around the bar. Clearly, this is physically rather strange behavior. For a more complete understanding, let's examine the solutions themselves, that is the angle versus time and angular velocity versus time plots. We have, respectively,

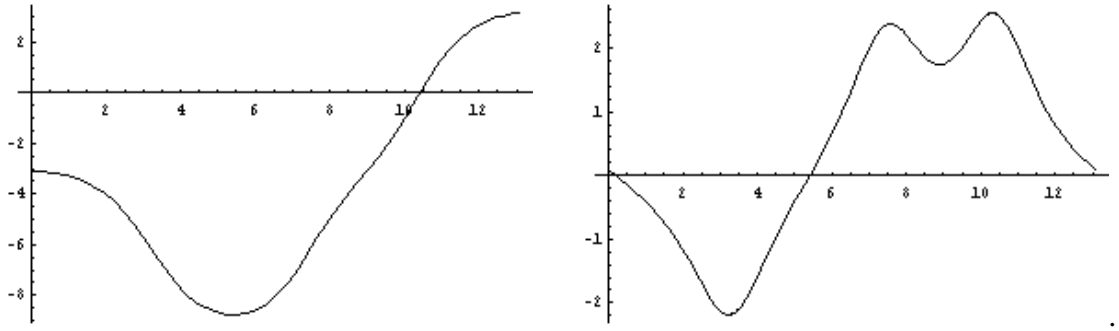


Figure 5.2a and 5.2b

As in the phase plane, we can read off the behavior of our gymnast—initial negative angular velocity implying clockwise motion, followed by a change in direction around

time $t = 5.5$ and a counterclockwise swing back to π. Note that these plots are not as "nice" as we might hope. They lack the smoothness and uniformity of our previous solutions—in a sense, the influence of the forcing function is more evident, as one might expect with the larger amplitude. As in the previous cases, we ought to consider the plot of the change in energy. We have,

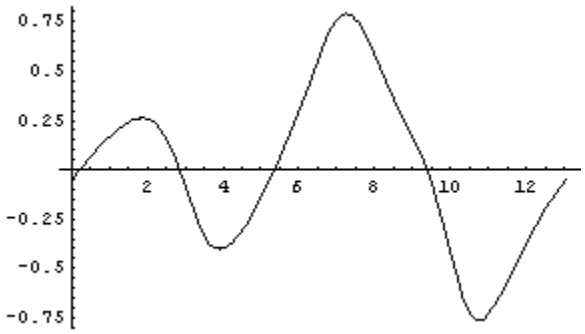


Figure 5.3

As we might expect by now, this plot is qualitatively different than our previous solution. There are far more regions of energy loss and gain here. Furthermore, our plot is not as symmetric as previous graphs. The last phase of the plot is an area of energy loss, indicated that the gymnast needed to slow down so as to not overshoot the endpoint. These differences in the energy function are reflective of the differences in overall solution behavior.

Section VI

Newton's Method: Another Application

Having found a way to a parameter value solution that satisfies our initial condition for a particular class of forcing functions, we might now attempt to bridge to a different forcing function. In fact, due to the close relationship between cosine and sine functions, it is reasonable to hope that starting from the solution for cosine forcing we might be able to step to a solution for sine forcing. Yet this will prove difficult. Nonetheless, we procure a starting solution for a sine forcing function, and using methods similar, though not identical, to those in the preceding section, we find a solution to our original initial value and endpoint value problem.

A Sine Forcing?

Recall our previous forcing function was of the form: $g(t) = A(Q\text{-}\cos(\omega t))$. Note the identity $-\cos(\omega t) = \sin(\omega t + \pi/2)$. Therefore, by starting with the forcing function $g(t) = A(Q\text{-}\cos(\omega t + R))$ with Q equal to zero (as solved previously) and stepping R from zero to $\pi/2$, we might hope to find a solution for a sine forcing function.

Unfortunately, such a method proves unsuccessful. Once R reaches approximately .15, it becomes increasingly difficult to find a solution. Newton's method quickly brings both parameters towards infinity, despite the use of scaling factors and very small step sizes. Why does it become so difficult to find solutions? Let's consider the physical implications of a shift from a cosine to sine function in this manner. In affect, we are altering the timing of our forcing function. One would expect there to be an ideal timing, thus it is not surprising that there is also "poor timing"—that is, we find R values such that finding a solution is difficult. By introducing this shift, it seems that between $R = 0$ and $R = \pi/2$ there is a string of values for which the timing of the forcing does not constructively work with the "natural swing." Thus it becomes very difficult to find parameter solutions for this range of R values. The complication prevents the implementation of this method of shifting our cosine forcing to a sine forcing. Therefore we must use more of a guesswork type of method in order to find a starting solution that will allow us to investigate the model under a sine forcing function.

In fact, this difficulty points us towards another point of consideration. As certain types of forcing seem more feasible than others, we might also consider the plot of change in energy over the course of our time interval. What pattern of change in energy leads to a successful giant swing? Are there multiple qualitatively different possibilities, or will most successful parameter values lead to a certain type of plot? Any such pattern could help us find parameter value solutions for other types of forcing. We ought to keep this question in mind while in search of solutions to our endpoint problem using a sine forcing function.

Sine Forcing

As our previous method proved unsuccessful, we start with our new forcing function itself. Here though, we chose slightly different roles for our parameters. We pick the forcing function $h(t) = \sin(\omega t + R)$. In this case, the frequency and the phase shift of our forcing are both altered through the parameters. We find ourselves a beginning solution through wise estimates for our initial parameter vector as well as different choices in initial and endpoint conditions. Once this solution is procured, we can step our initial and endpoint conditions towards the endpoint problem addressed in previous sections.

In fact, beginning with larger initial and endpoint velocity brings us away from the instability of the saddle point. As it may be easier to find solutions in this area using Newton's method, we start with the initial condition $(\theta_i, \dot{\theta}_i) = (-\pi, 1)$. Our endpoint condition is $(\theta_f, \dot{\theta}_f) = (\pi, .9)$. Again, we make things slightly easier by not demanding that all the initial velocity is recovered at the endpoint. We set the coefficient of friction, $b = .2$. Our starting parameter vector is $(R, \omega) = (2.94023, 1)$. Here, the R value is chosen such that the forcing at time zero is equal and opposite to the friction at time zero for original guess of parameters. That is, $h(0) = b\dot{\theta}(0)$ for our initial choice of parameters ω and R.

After a few iterations of Newton's method we have the result $(R, \omega) = (3.21651, 0.969452)$ which satisfies our desired endpoint condition. Let's consider a few properties of this solution. First, what does it look like in phase space? We have,

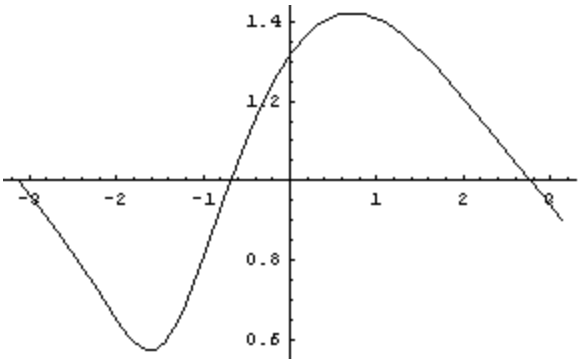


Figure 2.1

This does appear to be a fairly "nice" solution in its smoothness and general regularity. Yet it is interesting to note the initial fairly substantial decrease in angular velocity, a property not apparent in previous solutions. Though the angle continues to increase, the gymnast in fact slows down during the initial downward swing—a fairly odd result physically. But recall that here we start with a larger angular velocity than in previous cases.

The change in energy over time ought to be considered as well. We have the plot,

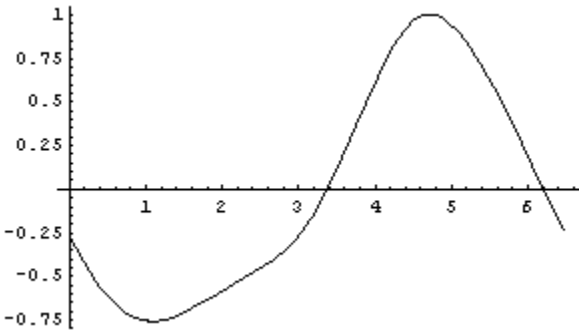


Figure 2.2

Consistent with our observation that the gymnast initially slows down, there is an initial energy loss here. Yet a large energy increase during the second half of the swing compensates for this and allows our gymnast reach to the desired endpoint condition. This serves as one possible plot shape for our change in energy over a successful giant swing. So we have found a initial parameter value solution from which we can hope to step towards a solution of our original initial value and endpoint problem.

Another Solution?

Before seeking a solution to our original question, we ought to take another look at the solution for the initial and endpoint value problem above. In fact, by starting from a slightly different initial parameter vector, we end up at different parameter vector

solutions that successfully satisfy our endpoint conditions. Two such solutions will be compared and discussed below.

First, we consider the solution given by the parameter values $(R,\omega) = (17.9061, -1.20104)$. Though one might initial expect these parameter values to produce the same forcing function over the time interval (since our forcing function is periodic—this solution could feasibly be just a shift of our original solution) this is not the case. This is perhaps immediately evident through considering the phase plane. We have,

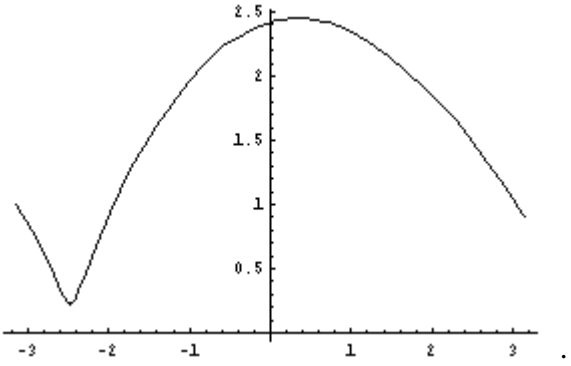
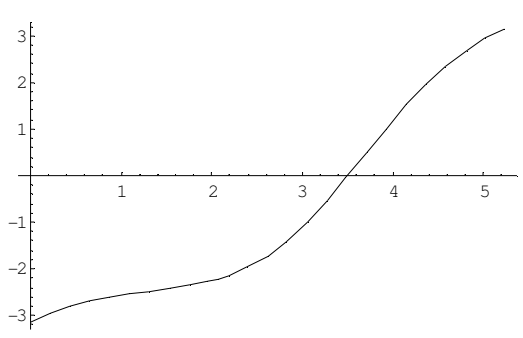


Figure 3.1

Clearly, the path of the solution here is not identical to the path dictated by the previous solution. While this solution could perhaps be considered a qualitative evolution of the previous shape, the distinctions are significant. Note that the initial concave up portion of this solution is considerably less substantial than the concave down portion, while the two were very similar in size for the other phase plane plot. For future reference, we also consider both of the solution curves over our time interval (angle versus time and angular velocity versus time respectively)



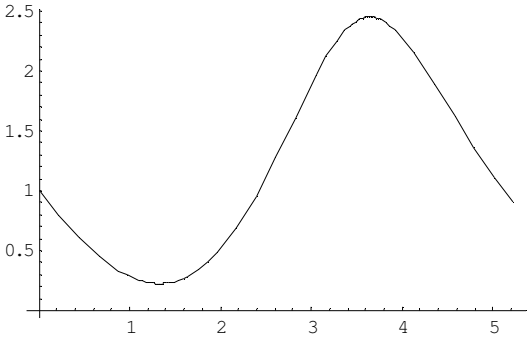


Figure 3.2a and 3.2b

It is interesting to observe that the second half of the swing occurs more quickly than the first half—evidence again of the influence of the forcing function. Also note the

inflection point on the angle versus time plot in the general area of the x-axis intercept. Yet even more interesting perhaps is the change in energy function. We have,

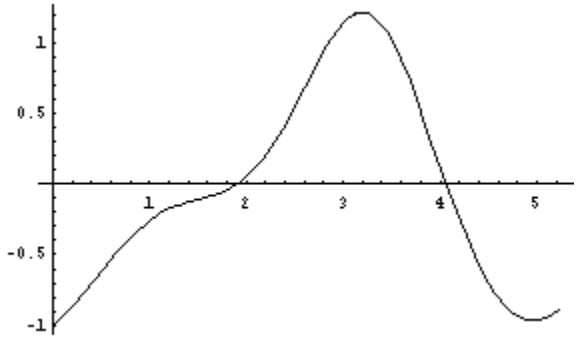


Figure 3.3

We start our trajectory with an energy decrease, followed by an energy increase, with its most substantial "kick" around time $t = 3.2$. This roughly corresponds to the bottom of the gymnast's swing (approximately $\theta = 0$). The gymnast then returns to the top of the bar and there is a decrease in energy as she needs to slow down to satisfy the endpoint condition. Note that the shape of the change in energy plot for this forcing is quite different than that of the other solution to this problem. The bulk of the energy increase comes at a different point in the swing. Furthermore, while the previous solution had one concave up region and one concave down region, here we see additional inflection points. Overall, observe that though the magnitudes are similar, the shape of the graphs are quite different.

By once again slightly altering our initial parameter vector, yet another solution can be found for this problem. A few iterations brings us to the solution $(R, \omega) = (5.36017, -1.08808)$. While these parameter values represent a forcing function different from the previous two, this solution exhibits behavior that is qualitatively similar to the $(R, \omega) = (17.9061, 1.20104)$ solution. That is, the change in energy plot and phase space plot are virtually identical. Yet the forcing functions represented by each set of parameters are different. Considering the difference of the two forcing functions, we have the plot,

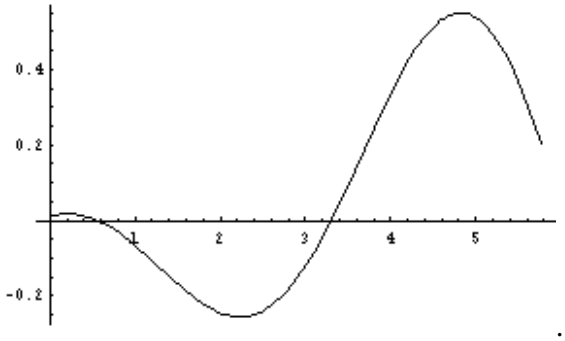


Figure 3.4

It seems that there is a substantial difference between the two forcing functions. Yet they both satisfy our endpoint condition and do so in a very similar manner—qualitatively the plots are the same. This is perhaps suggestive of a certain group of parameter values that satisfy our endpoint condition and do so in a similar manner despite differences in the actual forcing functions themselves.

So we have seen that first of all, given a particular endpoint condition and initial condition, there are a number of parameter value solutions for our forcing. Secondly, while some of these solutions will have qualitatively similar behaviors, some will also have qualitatively different behavior. If nothing else, this shows that there is not one unique change in energy curve that is qualitatively required for any successful giant swing. That is, a variety of techniques may allow for successful giant swings. We ought to keep both these points in mind over the course of our analysis.

Back to Our Problem

Let's return to our original question. We have (multiple) solutions for an initial and endpoint condition problem with our new forcing function. Yet the initial and endpoint condition problem that we have solved is not the same as our original question. Is there a way that we can use the solution we have to get a solution to the initial and endpoint condition problem we would like to solve?

We approach this question using a similar method as before. Previously we had a solution for a particular forcing function, then slowly changed the forcing function while maintaining contact with a solution. Here, we will slowly step down our initial and endpoint conditions, finding a new parameter solution for each step and using it as an initial parameter vector for Newton's method of the next step down until we reach the initial and endpoint conditions desired. In sum, our algorithm is:

1.) Begin with parameter vector solution, call it P_{s1}, for initial condition $(\theta,\dot{\theta}) = (-\pi,1)$ and end condition $(\theta,\dot{\theta}) = (\pi,.9)$.

2.) Consider the initial and endpoint value problem with initial condition $(\theta,\dot{\theta}) = (-\pi,.9)$ and end condition $(\theta,\dot{\theta}) = (\pi,.9)$.

3.) Take P_{s1} to be the initial parameter vector for Newton's method.

4.) Apply Newton's method, using a scaling factor as necessary, until a new solution parameter vector is reached. Call it P_{s2}.

5.) Consider the new initial value endpoint value problem with initial condition $(\theta,\dot{\theta}) = (-\pi,.8)$ and endpoint condition $(\theta,\dot{\theta}) = (\pi,.8)$. Take P_{s2} as the initial parameter vector for Newton's method and find a solution.

6.) Continue this process until a parameter vector solution for initial condition $(\theta,\dot{\theta}) = (-\pi,.1)$ and end condition $(\theta,\dot{\theta}) = (\pi,.1)$ is reached.

How successful is the implementation of this method? Can we in fact approach a solution in this manner or are we faced with problems similar to those encountered in trying to shift our cosine forcing to a sine forcing? Our first few steps work out rather well. The phase plane, change in energy function, and solution curves are qualitatively similar to those previously noted for the parameter vector *(R, ω) = (3.21651, 0.969452).* While it takes a number of iterations to reach the "close vicinity" of the parameter vector solution—that is, the range that allows Newton's method to converge quadratically—there is no need for the use of a scaling factor to reach this range. In fact, the process of Newton's method proceeds nicely until the jump from initial and endpoint condition angular velocity of .7 to .6. At this point, it appears that the solution to our previous system is not "close enough" to the actual solution of our system with adjusted parameter values. Hence Newton's method does not converge. Therefore, we decrease our step size from one solution to the next. By implementing smaller changes to the initial and endpoint angular velocity requirements, the previous solution ought to serve as a better starting parameter vector for Newton's method of the new initial and endpoint value problem. In fact, by shortening these step sizes, (to between .025 and .015) we continue to find parameter value solutions until we reach the initial and endpoint angular velocity condition .45. Taking the angular velocity from .5 to .45 proves quite problematic, even

using smaller step sizes. Hence, we choose to use a scaling factor of .1. Though the number of necessary iterations increases drastically here, we do eventually reach a solution $(R,\omega) = (3.04553, 0.474187)$ for the initial and endpoint angular velocity of .45 initial and endpoint condition. Yet qualitatively it seems that our solution has changed. As before, we can consider the phase plane, change in energy plot, and solution curves. The phase plane is as follows,

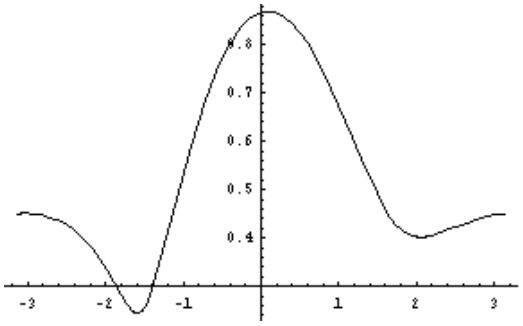


Figure 4.1

Note the dissimilarity to the original solution. This plot has a qualitatively very different shape. Perhaps most notable is the change in the tail end of the graph, with an increase in the angular velocity at the end of the swing. Not surprisingly, this parameter solution also leads to a different sort of change in energy function. We have,

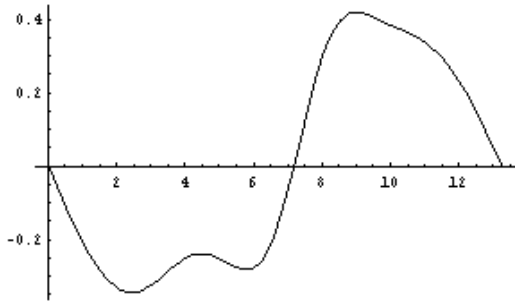


Figure 4.2

Again, though perhaps there still is a resemblance to our previous change in energy function, we also see qualitative differences. Note for example the additional local minimum point in the negative portion of the change in energy function. Generally, this seems to be a far less "regular" curve than our previous change in energy function. Finally, considering the solution curves, we have (angle and angular velocity respectively):

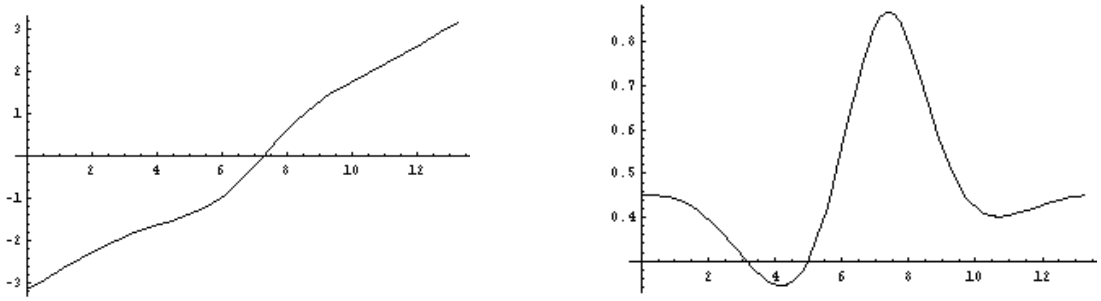


Figure 4.3a and 4.3b

These curves in fact bear some qualitative resemblance to those of the second solution (parameter vector *(R,ω) = (17.9061, -1.20104)*) found to the system with initial angular velocity 1 and endpoint angular velocity .9. We have the same continual increase in angle, though there appears to be hints of additional inflection points not seen in the previous solution. Accordingly, as one might expect, the angular velocity plots show more substantial differences. While the middle portion of the two plots indicates qualitatively similar behavior, the beginning and ending portions of this solution are concave down—behavior that was not present in our previous solution. So it seems that in reaching a parameter value solution for initial and endpoint angular velocity condition .45 we have moved to another type of solution curve. Perhaps this is unsurprising given the difficulty of finding a solution for this condition.

Carrying on the process of stepping down our initial and endpoint conditions while using a scaling factor in Newton's method, we can continue to produce solutions, though at times our step size again must be decreased in order for Newton's method to converge. But yet again, we observe a few instances of qualitative change in solutions. Consider the solution *(R,ω)=(3.07911, 0.310498)* found for the initial and endpoint angular velocity condition .3. Our phase plane is as follows,

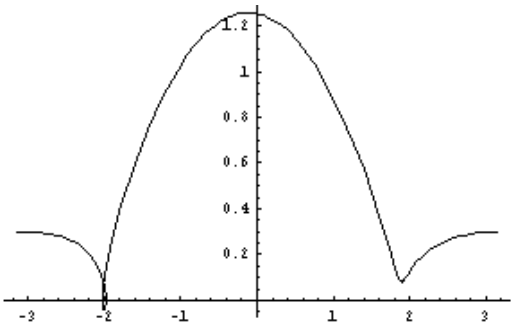


Figure 4.4

Quite a few items are of note here. First of all, observe that the center potion of our plot has become even more substantial in comparison to the two tail concave down areas. This is quite a transformation from our original solution, or even our more recent solutions. Second, note that the solution in the phase plane intersects itself at the θ value of approximately –2. That is, our solution passes through a particular position twice, indicating that the gymnast changes directions twice. This is certainly physically odd behavior for a giant swing. Considering the change in energy plot brings out further differences. We have,

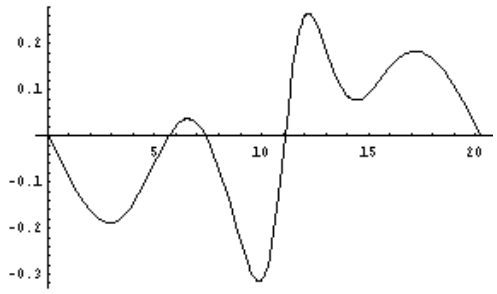


Figure 4.5

Again, there is a strong qualitative difference between this and previous plots. Here we have more intercepts and more local minimum and maximum values than any of our previous solutions. Instead of one region of energy increase and another of energy decrease, we have two of each. Additionally, the increase in local maximum and minimum hinted at in our previous solution (angular velocity of .45) has become even more extreme. For completeness and confirmation of the behavior noted in the phase plane, we ought to also consider the solution curves (angle and angular velocity respectively):

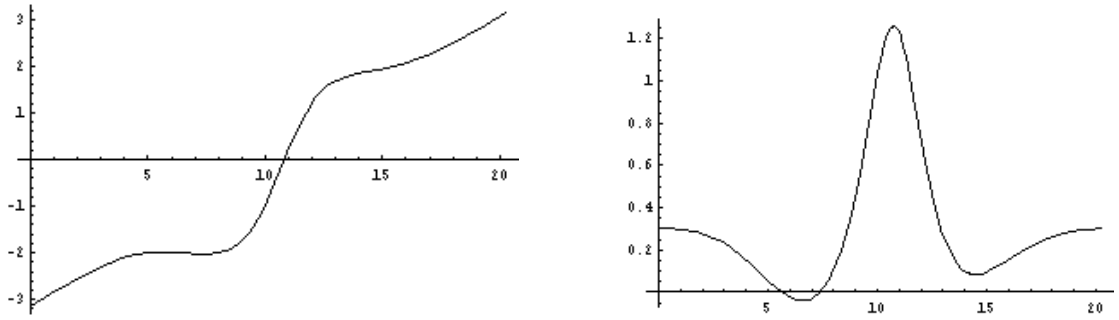


Figure 4.6a and 4.5b

Going along with previous observations, note the two time values for which our angle equals approximately –2. On the angular velocity plot, note the corresponding region of

negative angular velocity—that is, a region in which our gymnast swings clockwise. So it seems that we have another instance of a substantial change in solution behavior from one initial and endpoint angular velocity condition to the next. Also note that our solutions seem to be getting less physically reasonable—of particular concern is the gymnasts change in direction mid-swing.

Yet once again, another qualitative change in solution behavior occurs for initial and endpoint angular velocity condition of .225. Here we have parameter vector solution value, *(R, ω) = (2.26512, -0.31166).* Note that our ω value has become negative and gone through a fairly substantial change relative to its previous value of .254194. This change is reflected in our solutions. Our phase plane is given by,

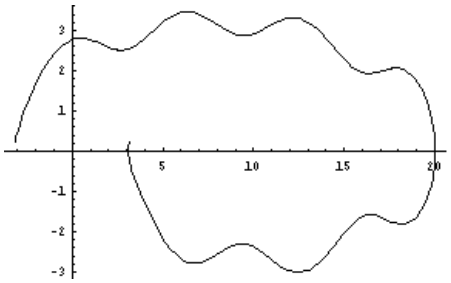


Figure 4.7

Clearly this solution is very different from our other solutions. Our gymnast travels counterclockwise around the bar multiple times, before traveling clockwise around the bar to, in a sense, "return to" an angle of π. This is perhaps not the behavior we envisioned in setting out this problem. Yet it is impressive that this forcing allows the gymnast to travel so many times around the bar over just one period of the forcing. Considering the change in energy plot, we have,

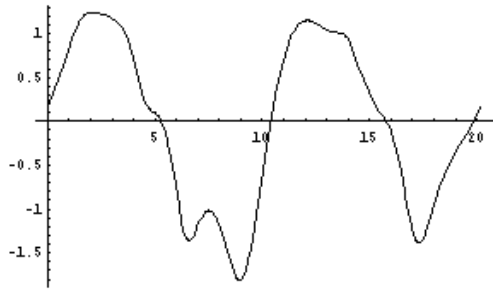
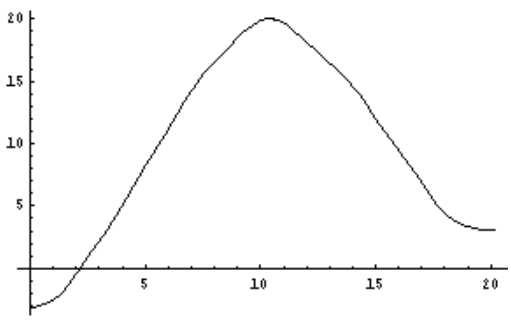


Figure 4.8

Again, we have a strong qualitative difference. The magnitudes of the minimum and maximum change in energy are once again similar, yet, the timing of the change in

energy has switched. That is, we start our swing by gaining energy and end our swing by losing energy. This difference is consistence with our initial substantial increase in angle (i.e. traveling around the bar counterclockwise multiple times). Finally, note the solution curves (angle and angular velocity):



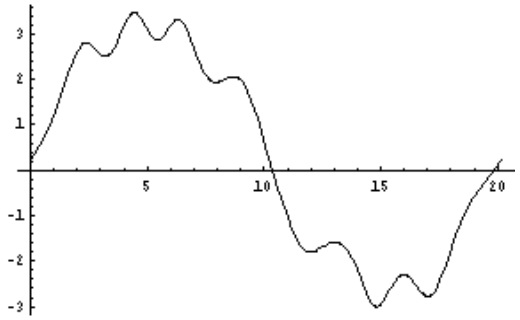


Figure 4.9a and 4.9b

As we would expect from the phase plane, the plot of the angle over time is qualitatively different from all other solutions—there is a large increase in angle, followed by a decreasing return to π. On the angular velocity plot, the areas of positive then negative angular velocity are clearly apparent.

Finally, we ought to consider our solution for initial and endpoint angular velocity condition .1. Stepping down these conditions and using a scaling factor as before, we end up with the parameter vector solution, $(R, \omega) = (2.32768, -0.31335)$. This solution is qualitatively identical to that above (that is, the solution for the initial and endpoint angular velocity of .225). Though our process has been successful in reaching a solution, we must note that it is not the solution we really had in mind in starting the problem—the gymnast completes numerous clockwise and counterclockwise giant circles before "returning" to the endpoint condition.

In this process we have come up with a variety of qualitatively unique solutions—sometimes to the same question. It seems that there is more than one manner to successfully complete a giant swing, as there are a variety of change in energy plots and paths through the phase plane that allow us to satisfy our initial and endpoint conditions. While considering the change in energy plot and phase plane is still instructive, it may not serve as a decisive manner of predicting the success or failure of a certain class of forcing functions.

Section VII

Having addressed at least a few possible forcing functions for the forced simple pendulum model we might now attempt a similar feat starting from our more complicated double pendulum model. Recall that here forcing was incorporated into the model through dictating the angle between the first and second rod of the pendulum. This simplified our system of four equations into a system of two non autonomous equations, though still a fairly complicated set of equations. As with our previous model, we would like to apply Newton's method to find parameter values within the forcing so as to satisfy our endpoint condition. Once again, our difficulty will come in finding a good initial guess for our parameter vector—one that is close enough to a solution (of which there will most likely be a few) to give us fast convergence.

Recall the Model

Briefly, let's recall our forced double pendulum with friction model. We have the system of equations,

$$\frac{d\theta}{dt} = \dot{\theta} \tag{1.1}$$

$$\frac{d\dot{\theta}}{dt} = \frac{-AS\omega^2 \sin(\omega t) + gK\sin(A\sin(\omega t) - \theta) - g(C+Hl)\sin(\theta) - AKl\omega(\omega\cos(\omega t)\cos(A\cos(\omega t)) + \sin(\omega t)\sin(A\cos(\omega t))(A\omega\sin(\omega t) + 2\dot{\theta}))}{D + Hl^2 + S - 2Kl\cos(A\cos(\omega t))} - b\dot{\theta} \tag{1.2}$$

Where our forcing function has been built in by dictating the angle between the two rods, that is, setting $\psi = \pi - \varepsilon \sin \omega t$ in our original four equation system. Thus our parameter values, to which we apply Newton's method, are ε and ω. We aim to solve a similar problem as before. We set the coefficient of friction to be $b = .05$ and all other parameters to one (excluding ε and ω of course). Our initial condition is $(\theta, \dot{\theta}) = (-\pi, .1)$ and our desired endpoint condition is $(\theta, \dot{\theta}) = (\pi, .1)$ at time, $t = \left|\frac{2\pi}{\omega}\right|$.

Theoretically Sound?

Before using Newton's method, we ought to consider whether our model satisfies the theoretical conditions necessary for Newton's method. The right hand side of our system of equations is analytic and thus differentiable and continuous. We need to take

caution though as there is the possibility of points of discontinuity and indiffertiability due to the denominator of the expression. We require that our denominator is nonzero for our acceleration expression to be defined. If $\left|\frac{D-Hl^2+S}{2Kl}\right|>1$ then the denominator is always nonzero, hence it would be wise to choose these parameter values so as to satisfy this condition. Unfortunately, original choices of parameters have this expression equal to one. Therefore there may be points of infinite acceleration, namely for t values, $t=\frac{\pi}{2\omega}+2\pi$. This observation is concerning, and may contribute to some of the difficulties encountered analyzing this system.

Newton's Method

In attempting to choose a good initial parameter vector guess for Newton's method, let's consider the physical significance of our forcing function. This function dictates the angle between the legs and torso of the gymnast throughout the swing. As previously noted, this implies a certain range of physically reasonable values for ε and ω. Physically, we would expect $-\pi/2 < \varepsilon < \pi/2$ and ω with "reasonably small" magnitude to allow a practical amount of time to complete the giant swing. Hence, we begin with the initial parameter vector (ε, ω) = (.2, .75). After a few iterations, Newton's method diverges. Thus we attempt to find a solution using the same initial parameter vector along with a scaling factor of .1. Under this method, we can get parameter values that are quite close to satisfying our endpoint condition, but do not seem to be converging to a parameter vector that will lead to a solution that actually satisfy our desired endpoint condition. After a point, our solution merely wanders slightly within a region, the error continually increasing a bit, then decreasing again, but never past a certain point. Yet these parameter values produce the general type of behavior we would like, unlike the physically unusual solutions found earlier. So instead of blindly searching for another reasonable beginning parameter vector, that might lead to a solution—but a solution with physically unreasonable behavior—let's stick with this region of parameter values.

Frustrated by our initial attempt, yet still dedicated to our initial parameter value guess, we attempt to make the giant swings "easier" by decreasing the coefficient of friction. Recall that this method was in fact successful for the initial forced pendulum

model. We take the coefficient of friction b = .000000005 and use our "almost a solution" parameter values *(ε, ω) = (1.47506, 1.70054)* from the larger friction problem as the starting parameter guess. Using Newton's method with scaling factor .1 until it appears that we are "close" to the solution, at which point the scaling factor is omitted (set equal to one), we end up with the parameter vector *(ε, ω) = (1.50542, 1.81611).* Yet this is still not a solution for our problem, as there is a continuous lingering endpoint error of magnitude approximately 1.4×10^{-5} on θ and 5.1×10^{-7} on $\dot{\theta}$. This is fairly unsatisfying; it seems that there may not be an actual solution in this vicinity. Nonetheless, let's consider the phase plane for this solution that almost satisfies our endpoint condition. We have

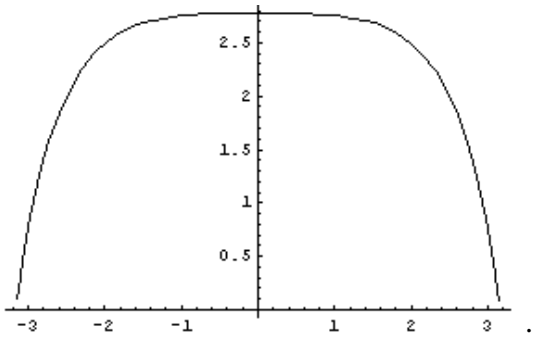


Figure 2.1

Note that, other than the fact that it does not strictly satisfy our endpoint condition, this is a quite nice solution curve. Our gymnast travels around the bar almost exactly once with a regular increase in velocity, with maximum velocity at the bottom of the swing ($\theta = 0$) followed by a smooth decrease in velocity as θ increases back up to (almost) π. Considering the change in energy function, we have,

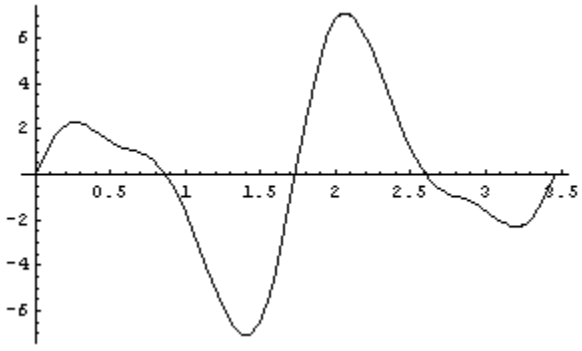


Figure 2.2

While perhaps this is less idealized, it is a familiar qualitative curve. We have two regions of energy increase and two regions of energy decrease and the plot has a sort of

symmetry. This recalls the change in energy functions of previous successful solutions for our forced single pendulum with friction system.

No Friction

In a bold effort to reach a solution, we consider our model with friction equal to zero—that is, we remove the constantly dissipative term in the system. In this instance can we reach an actual solution for this parameter range? In fact, our efforts are foiled again. While we reach parameter values that get us even closer to the endpoint solution than previous efforts, there is still no indication of convergence to a solution. The error continues to fluctuate. Additionally, we get the impression of bouncing back and forth between a few "almost" solutions in the same region. From time to time our parameter vector solution undergoes a more substantial shift (plus or minus approximately .2 for each parameter) and the error temporarily increases, before decreasing again for a slightly different parameter vector than our prior "almost" solution. It seems that we have an area in which many of our parameter vectors put us very near, but not at, our endpoint condition. The phase plane and change in energy plots for these "almost" solutions are qualitatively the same as our "almost" solutions above.

The Search for a Solution Continues: Negative Coefficient of Friction

As a last ditch attempt, let's consider a negative coefficient of friction. That is, we have really changed the identity of the "frictional term" so that it now continuously provides energy to the system. Physically this makes little sense, though it might be argued that this is in fact similar to the manner in which we tacked on a forcing function to the simple pendulum model. So, in a sense, we have two forcing functions now that are potential sources of energy gain.

Unfortunately, this does not solve our difficulties. We are presented with behavior similar to that above. This is in fact intriguing. One of our modes of making giant swings "easier" has not been effective in helping us approach a solution. What other techniques might be of service here? Increasing the initial angular velocity is equally ineffective. Other possibilities include altering the endpoint angular velocity condition or using a parameter phase shift in the forcing function, as was introduced in the second forcing function for the simple pendulum.

In Sum

Though we were unsuccessful in finding a solution for initial and endpoint condition problem applied to our forced double pendulum with friction model we have still noted some key behaviors of this system. Furthermore, we have multiple solutions for our forced single pendulum model (a system which is physically reasonable in itself) under various forcing functions. The double pendulum model leaves us with an intriguing question and some possible methods from which to approach it. It is likely that finding a solution to this problem is best achieved through a variation on these methods.

Conclusion

While this study has provided an introductory basis for discussion, perhaps more than anything it has pointed to the many intriguing questions that remain unanswered. Briefly we will bring up some of these issues as possible areas of further study.

Most obvious perhaps is the issue of finding a solution to our initial and endpoint condition problem for our more complicated model. While our "almost" solutions seemed to behave similarly to the solutions found for our single pendulum model, which is telling in itself, we would like to reach an actual solution. Ideally, we would also be able to locate some physically strange behavior solutions as well, as a point of comparison to our simple forced pendulum.

Second, one of the more significant difficulties encountered in this study was finding a reasonable initial parameter vector—one that was close enough to a solution that we could expect to reach a solution using Newton's method. In many cases, our initial parameter vector had to be quite close (each component within 0.01) of our parameter vector solution in order for us to converge to a solution. This presents a substantial difficulty. Thus we would like some way of globally searching to find parameter value regions from which we can pick an initial parameter vector and expect to converge to a solution.

Additionally, we would like to be able to compare our two models—is our more complicated model actually superior to our forced single pendulum model? Does it provide a better representation of the reality of giant swings? Even within each model, we would like to find the "ideal" solution—that is, a forcing function that most efficiently allows a complete giant swing. While the idea of "most efficient" needs to be further defined, we might think of it as something along the lines of a swing with a minimum value of the magnitude of dH/dt. This question can also be considered relative to the forcing functions. We have discussed only a small class—though perhaps one of the most physically reasonable classes—of forcing functions and the behavior of our system under other forcing functions is a topic of interest as well.

In terms of the theory behind Newton's method, we would like to have a better understanding of the convergence of Newton's method when a scaling factor is used.

Does Newton's method still converge and if it does, at what sort of rate does it converge at? Furthermore, the affects of chaos in conjunction with Newton's method ought to be more thoroughly investigated. Though certainly the behavior of Newton's method seems to suggest an increased difficulty due to a potentially chaotic system, this point ought to be theoretically developed. Furthermore, one might employ the variety of tools in chaotic dynamics in order to come up with some sort of measure of the chaos of the systems we are dealing with. Finally, we would like to be able to make a statement about the number of distinct solutions we would expect for each particular forcing function and model. This knowledge provides information on the physical significance of our model as well as helping focus our search for solutions.

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